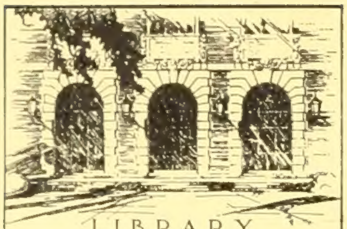


HIGH SCHOOL MATHEMATICS

Teachers' Edition

UNIT 5

UNIVERSITY OF ILLINOIS COMMITTEE ON SCHOOL MATHEMATICS

MAX BEBERMAN, *Director*

HERBERT E. VAUGHAN, *Editor*

UNIVERSITY OF ILLINOIS PRESS · URBANA, 1960

TEACHERS COMMENTARY

The introduction to this unit [pages 5-A through 5-K] is designed to create a nonverbal awareness of what a relation is. The student is expected to learn, at least informally, that a relation is a set of ordered pairs. The definition of a relation as a set of ordered pairs is given explicitly on page 5-1. There follows a section on the algebra of sets [unioning and intersecting sets are somewhat like adding and multiplying numbers], and another which contains illustrations of relations which arise from geometric problems. Section 5.04 introduces the notions of domain, range, and converse of a relation and discusses the properties of reflexivity and symmetry. Functions are relations of a special kind and are discussed in section 5.05. The remainder of the unit deals with related matters: variable quantities, functional dependence [including variation], linear functions, quadratic functions, and systems of equations.

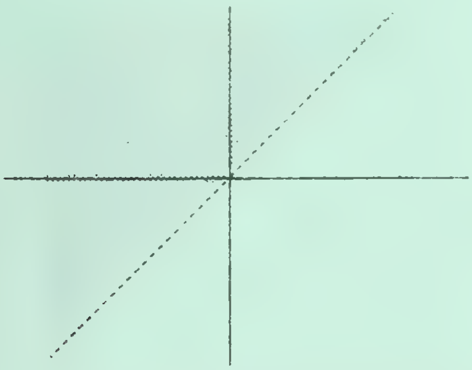
*

Each student should be supplied with [or supply himself with] a ruler, a protractor, a compass, and cross-section paper [4(or 5)-to-the-inch and 10-to-the-inch varieties]. He will also need "lattice paper" which is not obtainable commercially. [4-to-the-inch cross-section paper can be used for this purpose.] Lattice paper is easy to manufacture if you have a mimeograph machine or spirit duplicator available. Make some of the type shown on page 5-B and some like that on page 5-40. Students should keep a supply of cross-section paper and lattice paper in their notebooks. [It may be helpful to have a three-hole punch in the classroom, so that the paper can be readily stored in the notebook.]

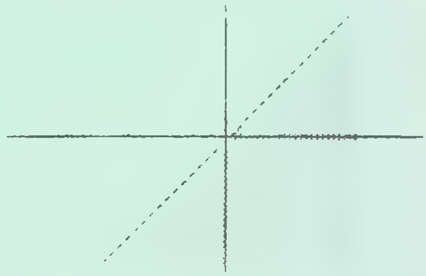
You should also have readily available a cross-section blackboard and a lattice blackboard so that you need not waste time when giving blackboard illustrations. Conditions vary too much from classroom to classroom for us to make specific suggestions which are universally applicable. One general suggestion is that you give prior thought to this problem and consult school supply guides.

Teachers who have used earlier editions of Unit 5 should especially note that the rules for games like TREE which are used in the present edition are different from those used in earlier editions. As stated on TC[5-A, B, C, D, E], when cards with values x and y are played, in that order, the trick belongs to the second player if and only if the graph of (x, y) is one of the heavy dots on the chart. [In earlier editions the trick belonged, in this case, to the first player.] This change has the effect that, for example, the sentence '$(x, y) \in T$' is equivalent to 'y TREES x' [rather than, as previously, to 'x TREES y']. It is believed that this will make easier the passage from the consideration of relations in general to that of functions.

This change also affects some of the work on relations in sections 5.01, 5.03, and 5.04. For example, under the new convention, y has the relation greater-than to x [*cf.* 'y TREES x'] if and only if (x, y) belongs to the relation greater-than. So, the relation greater-than is $\{(x, y): y > x\}$, and its graph is:



With the convention used in earlier editions the graph is:



For another example [although the definitions of domain and range of a relation are the same as previously], in the discussion of the relation of being-an-uncle-of in section 5.04, the domain of this relation is now the set of all nephews and nieces, and the range is the set of all uncles.

Correction. In the line just below the picture of Morris' cards, the sentence should read:

Since the deck is made up of ---

Students may find it helpful to make a copy of the chart on page 5-B. This will save a good deal of page flipping as they work from page 5-C through page 5-K. A copy of the chart on the blackboard for class discussion may also help.

*

On page 5-A, Tony's cards should be marked with these numerals:

3, 9, 7, 10, 6, 5, 4, 2, 8, 5

Morris thinks he has won the 3, 4-trick because he believes that, as in most card games, the higher card of the two played wins the trick. You may wish to have the students construct decks of twenty cards [use 3 × 5 index cards] like that described on page 5-A and actually play one or two games of TREE or UPPER TRIANGLE [page 5-J]. Be careful not to overdo this. The only value in such activity is the help it gives students in understanding the procedure for determining the winner of a trick. When cards with values x and y are played, in that order, the trick belongs to the second player if and only if the graph of (x, y) is one of the heavy dots on the chart. [Of course, another way of looking at it is to notice that the trick belongs to the first player if and only if the graph of (x, y) is one of the light dots on the chart.]

*

Answers for Part A [on page 5-E].

Morris first: 2nd card wins because 5 TREES 1.

Tony first: 2nd card wins because 1 TREES 5.

Morris first: 1st card wins because 10 DOES NOT TREE 2.

Answers for Part B.

1. T 2. T 3. F 4. T 5. F 6. F 7. T

8. T 9. T 10. T 11. F 12. F 13. T

As class discussion exercises, try these:

If you want to win the trick, what are the best cards to have in your hand when

(a) it is your turn to play second? [Answer: 4 and 5]

(b) it is your turn to play first? [Answer: 1 and 10]

If you want to win the trick, what are the worst cards to have in your hand when

(a) it is your turn to play second? [Answer: 2, 3, 9, and 10]

(b) it is your turn to play first? [Answer: 5 and 6]

*

Words like 'lattice' and 'cartesian square' were used in Unit 4. [See pages 4-E ff., section 4.01, and the related COMMENTARY.] In making the operation sign for cartesian product, students should make a large boldface times sign to distinguish it from the ordinary times sign. [Read '$D \times D$' as 'D cross D'.] It may be helpful to insert here a few exercises for class discussion such as:

List the ordered pairs in the cartesian square of $\{1, 2, 3\}$. [Answer: (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)]

Does (4, 3) belong to the cartesian square of $\{3, 4, 5, 6\}$? [Answer: yes]

How many ordered pairs are there in $C \times C$ where C is the set of integers from 1 through 100? [Answer: 10,000]

List the ordered pairs in the intersection of the cartesian square of $\{1, 2, 3, 4, 5\}$ and the cartesian square of $\{3, 4, 5, 6\}$. [Answer: (3, 3), (3, 4), (3, 5), (4, 3), (4, 4), (4, 5), (5, 3), (5, 4), (5, 5)]

[One occasionally sees an exponent symbol used in denoting a cartesian square. For example, 'D^2' may be used as an abbreviation for '$D \times D$'.]

$*$

Notice that, in order to avoid ambiguity, we speak of the complement of T with respect to $D \times D$. When referring to the complement of a set, one does so always with respect to some containing set. In each instance, the containing set consists of those objects with which we are then concerned, and is often called 'the space'. At present the space in which we are interested is $D \times D$. When it is clear what the space is, one is justified in not referring to it. So, we can here abbreviate 'the complement of T with respect to $D \times D$' to 'the complement of T' and abbreviate the latter to '$\widetilde{T}$'. Note that, with this convention as to the meaning of '$\widetilde{T}$', (7, 16) belongs neither to T nor to $\widetilde{T}$. However,

$$\forall_{x \in D} \forall_{y \in D} [(x, y) \in T \text{ or } (x, y) \in \widetilde{T}].$$

In general, for each space S and each set $E \subseteq S$, the complement of E with respect to S [or: $\widetilde{E}$] is $\{x \in S : x \notin E\}$. So, $E \cup \widetilde{E} = S$ and $E \cap \widetilde{E} = \emptyset$.

[In some texts, the symbol '$\widetilde{B}_A$' is used to refer to the complement of B with respect to a containing set A. In others, the symbol '$A - B$' is used for the same purpose.].

$*$

A bit of imagination may help in getting across the notion of complement. Ask students to imagine that $D \times D$ is pictured by a 10-by-10 array of tiny light bulbs. Supply current to some of these bulbs, and you get a light picture of T. Divert the current from these bulbs to other bulbs in $D \times D$, and you get a light picture of $\widetilde{T}$. [See, also, TC[5-K]a.]

In the last four lines on page 5-F, students are asked whether they see that the sentences '4 DOES NOT TREE 9' and '$(9, 4) \in \widetilde{T}$' are equivalent. They may not! The sentences <u>are</u> equivalent because when one says that 4 DOES NOT TREE 9 one means that the graph of (9, 4) is not one of the heavy dots on the chart; that is, one means that $(9, 4) \in \widetilde{T}$. So, '4 DOES NOT TREE 9' and $(9, 4) \in \widetilde{T}$' say the same thing.

Now, it happens that $(9, 4) \in T$. So, the sentences:

4 DOES NOT TREE 9 and: $(9, 4) \in T$

are both false. The sentences '4 TREES 9' and '$(9, 4) \in T$' are equivalent, and true. Also, the sentences '9 DOES NOT TREE 4' and '$(4, 9) \in \widetilde{T}$' are equivalent, and true.

*

To help students understand the idea of the complement of a set with respect to a containing space, try exercises like these in class. [Perhaps you can duplicate such a list in advance and pass it out to the students at this time.]

(1) Suppose the space is A where $A = \{1, 2, 3, 4, 5\}$. If B is $\{1, 2, 5\}$, what is $\widetilde{B}$? [Answer: $\widetilde{B} = \{3, 4\}$]

(2) Suppose the space is A where $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. If $B = \{1, 2, 5\}$, what is $\widetilde{B}$? [Answer: $\widetilde{B} = \{3, 4, 6, 7, 8, 9, 10\}$]

(3) Suppose the space consists of the students in your school. What is the complement of the set of students in your mathematics class? [Answer: the set consisting of those students who are <u>not</u> in the mathematics class]

(4) Suppose the space consists of the students in the classroom. What is the complement of the set consisting of those students who do not sit in the second row? [Answer: the set consisting of those students who <u>do</u> sit in the second row]

(5) Suppose the space is $S \times S$ where $S = \{1, 2, 3, 4, 5\}$. If R is the cartesian square of $\{2, 3, 4, 5\}$, what is $\widetilde{R}$? [Answer: $\widetilde{R} = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (3, 1), (4, 1), (5, 1)\}$]

(6) Suppose the space is G, where $G = \{(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4), (4, 2), (4, 3), (4, 4)\}$. If $H = \{(2, 2), (3, 2), (2, 4), (4, 2), (4, 4)\}$, what is $\widetilde{H}$? [Answer: $\widetilde{H} = \{(2, 3), (3, 3), (3, 4), (4, 3)\}$]

(7) Suppose the space is A, where A is the cartesian square of $\{a, e, u\}$. If $V = \{(a, a), (a, e), (a, u), (e, a), (u, a), (u, u)\}$ what is $\widetilde{V}$? [Answer: $\widetilde{V} = \{(e, e), (e, u), (u, e)\}$]

(8) Suppose the space is the number plane. What is the complement of the set of all ordered pairs of real numbers such that the second component is not equal to the first component? [Answer: the set of all ordered pairs of real numbers such that the second component <u>is</u> equal to the first component. Using brace-notation, we would write:

$$\widetilde{\{(x, y): y \neq x\}} = \{(x, y): y = x\}]$$

(9) Suppose the space is the number plane. What is the complement of the first quadrant? [Answer: the set consisting of the ordered pairs of real numbers in any of quadrants II, III, IV, or in either of the axes. In other words:

$$\widetilde{\{(x, y): x > 0 \text{ and } y > 0\}} = \{(x, y): x \leq 0 \text{ or } y \leq 0\}$$

Recall that the domain of 'x' and 'y' is the set of real numbers.]

(10) Suppose the space is the number plane, and $A = \{(x, y): xy = 0\}$. What is the complement of A? [Answer: $\widetilde{A}$ consists of those ordered pairs of real numbers that are not in either of the axes. A shorter way of giving the answer is: $\widetilde{A} = \{(x, y): xy \neq 0\}$]

*

Exercises 9 through 14 may be done in at least two ways. The student might just try replacements for 'x' which will convert the sentence into a true sentence. A more efficient way to do Exercise 9, for example, would be to graph S, the set of ordered pairs in $D \times D$ whose second components are twice their first components. The solution set of '$(x, 2x) \in T$' is $S \cap T$. A similar procedure could be used in Exercises 10-14.

*

For Exercises 15 and 16, note that an ordered pair is in $\approx T$ just if it is a point of $D \times D$ which is not in $\tilde{T}$. And, since a point is in $\tilde{T}$ if and only if it is a point of $D \times D$ which is not in T, a point of $D \times D$ is not in $\tilde{T}$ if and only if it is in T. So, an ordered pair is in $\approx T$ if and only if it is in T. In other words, $\approx T = T$.

*

Skill quizzes, and quizzes on content, are given on the pages listed below.

TC[5-I, J]b
TC[5-18]b
TC[5-39]c
TC[5-71]
TC[5-96]d, e
TC[5-122]
TC[5-144, 145, 146]c
TC[5-163, 164, 165]b
TC[5-189, 190]b
TC[5-217, 218]b
TC[5-K]e, f
TC[5-23]ℓ
TC[5-47]d
TC[5-78]
TC[5-106]b, c
TC[5-133]b
TC[5-148, 149]b
TC[5-169]b
TC[5-197]c
TC[5-12]e
TC[5-37]c
TC[5-55, 56]c
TC[5-85]
TC[5-115]b, c
TC[5-138, 139]b
TC[5-156, 157]b
TC[5-183]b
TC[5-207, 208]b

A comprehensive examination over pages 5-A through 5-115 is given on TC[5-117, 118]b, c, d, e, f, g.

A collection of quiz items for all of Unit 5 is given on TC[5-218]a, b, c, d, e, f, g, h, i .

The use of 'ϵ' as an abbreviation for 'is an element of' was introduced on TC[3-112]a. 'ϵ' may also be read as 'is a member of' or as 'belongs to'.

*

Answers for Part C.

1. F

2. T [After the student has answered 'false' to Exercise 1, he should immediately know that the answer for Exercise 2 is 'true'. Similarly for Exercises 3 and 4.]

3. T 4. F 5. F 6. T 7. F

8. F 9. T 10. T 11. T 12. F

*

In Part D the student is asked to find the solution set for each sentence, that is, to find values of 'x' which satisfy the sentence. The domain of 'x' is D.

*

Answers for Part D.

1. $\{5, 6\}$

2. $\{1, 2, 3, 4, 7, 8, 9, 10\}$, or: $\widetilde{\{5, 6\}}$

   [When the answer is given as '$\widetilde{\{5, 6\}}$', it must be understood that the set named is the complement of $\{5, 6\}$ <u>with respect to D</u>.]

3. $\widetilde{\{1, 10\}}$, or: $\{2, 3, 4, 5, 6, 7, 8, 9\}$

4. $\{1, 10\}$, or: $\widetilde{\{2, 3, 4, 5, 6, 7, 8, 9\}}$ [Note that once a student has answered Exercise 3, he should be able to answer Exercise 4 immediately.]

5. $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, or: D, or: $\widetilde{\emptyset}$

6. $\emptyset$, or: $\widetilde{D}$

7. $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, or: D, or: $\widetilde{\emptyset}$

8. $\{1, 4, 5, 6, 7, 8\}$ 9. $\{2, 3, 4, 5\}$ 10. $\{3, 4, 5\}$

11. $\{5, 6, 7, 8\}$ 12. $\{1, 7, 8\}$ 13. $\{3, 4, 5\}$

14. $\{2, 3, 5, 6\}$ 15. $\{3, 4, 5, 6, 7, 8\}$ 16. $\{4, 5\}$

Correction. In Exercise 3, change '$\subset$' to '$\subseteq$'.

You [or your students] may wonder why we have written:

(1) $\{x \in D:\ (x, 3) \in T\}$

rather than merely:

(2) $\{x:\ (x, 3) \in T\}$

[In Unit 4, pages 4-9 ff., we introduced restrictions similar to '$\in D$'. There we wrote, for example, '$\{(x, y),$ x and y integers: $x = y - 13\}$'. In this unit we shall write, instead of this, '$\{(x, y) \in I \times I:\ x = y - 13\}$', using 'I' as a name for the set of integers.] Such names as (1) have the advantage that, for example,

$$\overline{\{x \in D:\ (x, 3) \in T\}} = \{x \in D:\ (x, 3) \notin T\}.$$

Thus, complementation with respect to the space D is associated with <u>denial</u>, as expressed by the slash ['$\notin$'], just as union is associated with <u>alternation</u>, as expressed by 'or':

$$\{x \in D:\ (x, 3) \in T\} \cup \{x \in D:\ (x, 5) \in T\} = \{x \in D:\ (x, 3) \in T \text{ or } (x, 5) \in T\}$$

and intersection is associated with conjunction, as expressed by 'and':

$$\{x \in D:\ (x, 3) \in T\} \cap \{x \in D:\ (x, 5) \in T\} = \{x \in D:\ (x, 3) \in T \text{ and } (x, 5) \in T\}$$

In contrast, $\{x:\ (x, 3) \notin T\}$ consists of all real numbers which do not belong to D as well as those which belong to $\{x \in D:\ (x, 3) \notin T\}$.

The same result might have been obtained, in this case, by writing

(3) $\{x:\ x \in D \text{ and } (x, 3) \in T\}$

instead of (1). However, (1) has the advantage that it <u>first</u> points out to the reader what elements come into the total picture [that is, the members of the space under consideration], and it <u>then</u> tells him how to select from these the members of the subset which it names.

Also, as has been pointed out on TC[3-27]c, restrictions like '$\in D$' are sometimes necessary if one is to avoid nonsense. For example, '$\{x:\ x/x = 1\}$' is unsatisfactory because the result of substituting '0' for 'x' in '$x/x = 1$' is neither true nor false. Instead, it is meaningless. So, we write '$\{x \neq 0:\ x/x = 1\}$' to indicate that our usual convention that the domain of 'x' is the set of real numbers is here replaced by the convention that the domain of 'x' is the set of nonzero real numbers. Similarly, '$\{(x, y):\ x = \sqrt{y}\}$' is unsatisfactory because, although -3, for example, belongs to the domain of 'y', '$\sqrt{-3}$' is nonsense [when, as now, we are dealing with real numbers]. So, we write '$\{(x, y),\ y \geq 0:\ x = \sqrt{y}\}$', instead. [Read this as 'the set of (x, y), y nonnegative, such that $x = \sqrt{y}$'.]

*

Samples 1 and 2 of Part E provide the students with a review of the notions of subset and equality of sets. For each set A, for each set B, A = B if and only if each member of A is a member of B, and each member of B is also a member of A. Hence [Sample 1], in order to show that $\{x \in D: (x, 3) \in T\} \neq \{x \in D: (x, 8) \in T\}$, it is sufficient either to find at least one member of $\{x \in D: (x, 3) \in T\}$ which is not a member of $\{x \in D: (x, 8) \in T\}$, or to find one member of $\{x \in D: (x, 8) \in T\}$ which is not a member of $\{x \in D: (x, 3) \in T\}$.

Sample 2 deals with the notion of subset. For each set A, for each set B, $A \subseteq B$ if and only if each member of A is a member of B. To show that $A \not\subseteq B$, we must find at least one member of A which is not a member of B. Since there is no member of $\{5, 6\}$ which is not a member of $\{4, 5, 6, 7\}$ [alternatively: since each member of A is a member of B], $\{5, 6\} \subseteq \{4, 5, 6, 7\}$. This method of testing to decide if a first set is a subset of a second set is helpful in showing that the empty set is a subset of every set. Since, by definition, the empty set contains no members, there is no member of the empty set which is not a member of any given set. Hence, the empty set is a subset of every set. [See Exercise 10 of Part E on page 5-I.]

*

Here is a summary of some of the properties of the empty set. For each set A, $\emptyset \subseteq A$, $A \times \emptyset = \emptyset = \emptyset \times A$, and [for each set B] if $A \times B = \emptyset$ then $A = \emptyset$ or $B = \emptyset$. Also, $A \cup \emptyset = A$, $A \cap \emptyset = \emptyset$, and, with respect to any space S, $\widetilde{\emptyset} = S$ and $\widetilde{S} = \emptyset$.

*

Note that by mentioning subsets we can shorten the description of equality of sets. Thus, for each set A, for each set B, A = B if and only if $A \subseteq B$ and $B \subseteq A$.

*

Some authors use '$\subset$' for 'is a subset of', rather than '$\subseteq$'. We prefer to reserve '$\subset$' for 'is a proper subset of'. So, for each set A, for each set B, $A \subset B$ if and only if $A \subseteq B$ and $A \neq B$. Note the analogy between $\subset$, $\subseteq$ and $<$, $\leq$.

*

Answers for Part E [on pages 5-H and 5-I].

1. T
2. F
3. T

Quiz.

Consider the game, CROSS, played with the same cards as TREE. The chart used for CROSS is pictured below on a 10-by-10 lattice, where $D = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

Second										
10	×	×	•	•	•	•	•	•	×	×
9	×	×	×	•	•	•	•	×	×	×
8	•	×	×	×	•	•	×	×	×	•
7	•	•	×	×	×	×	×	×	•	•
6	•	•	•	×	×	×	×	•	•	•
5	•	•	•	×	×	×	×	•	•	•
4	•	•	×	×	×	×	×	×	•	•
3	•	×	×	×	•	•	×	×	×	•
2	×	×	×	•	•	•	•	×	×	×
1	×	×	•	•	•	•	•	•	×	×
	1	2	3	4	5	6	7	8	9	10

First

A. True or False?

1. $\{x \in D: (x, 5) \in C\} \subseteq \{x \in D: (x, 7) \in C\}$
2. $\{x \in D: (4, x) \in C\} = \{x \in D: (x, 4) \in C\}$
3. $\{x \in D: (1, x) \notin C\} \subseteq \{x \in D: (9, x) \notin C\}$
4. $\{x \in D: (x, 2) \in \widetilde{C}\} \subseteq \{x \in D: (x, 10) \in \widetilde{C}\}$
5. $\{x \in D: (2, x) \in \widetilde{C}\} = \{x \in D: (x, 6) \notin C\}$

B. Simplify.

1. $\{x \in D: (x, 10) \in C\} \cap \{x \in D: (x, 3) \in C\}$
2. $\{x \in D: (x, 8) \notin C\} \cup \{x \in D: (9, x) \in C\}$
3. $\{x \in D: (5, x) \in \widetilde{C}\} \cap \{x \in D: (x, 2) \in C\}$
4. $\{x \in D: (4, x) \in C\} \cap \{x \in D: (x, 10) \in C\}$
5. $\{x \in D: (x, 7) \notin \widetilde{C}\} \cup \{x \in D: (1, x) \in C\}$

*

Answers for Quiz.

A. 1. T 2. T 3. F 4. T 5. F

B. 1. $\{2, 9\}$ 2. $\widetilde{\{4, 7\}}$ 3. $\widetilde{\{4, 5, 6, 7\}}$ 4. $\emptyset$ 5. $\widetilde{\emptyset}$

> **Correction.** In the Solution for Sample 1, the
>
> second line should read:
>
> that is, we want to know what elements
> they have in

4. T 5. T 6. T 7. T 8. T 9. T 10. T

*

Answers for Part F.

1. $\widetilde{\{1, 2, 9, 10\}}$ or $\{3, 4, 5, 6, 7, 8\}$ 2. D 3. $\{4, 5, 6\}$

4. $\widetilde{\{4, 5, 6\}}$ 5. $\{5\}$ 6. $\widetilde{\{1, 8\}}$ 7. $\emptyset$ 8. $\emptyset$

9. $\widetilde{\{1, 10\}}$ 10. $\{4, 5, 6\}$

*

In solving Exercises 3 and 4 of Part F students may discover that for each set A, for each set B, $\tilde{A} \cup \tilde{B} = \widetilde{A \cap B}$. This is a theorem from the algebra of sets, and is one of two such theorems which are called 'DeMorgan's Laws'. Other theorems can be obtained from this by interchanging '$\cup$' and '$\cap$'. Other theorems of the algebra of sets are discussed in section 5.02. [Augustus DeMorgan was a nineteenth century mathematician who made important contributions to logic.]

*

UPPER TRIANGLE [page 5-J] is played with the same cards as TREE, and also with the same type of rule. That is, the second card wins the trick just if the ordered pair of cards corresponds to a heavy dot in the chart for UPPER TRIANGLE. [If each player plays a 3, the first player wins.]

*

Answers for Part G [on page 5-J].

1. $\{1, 2\}$ 2. $\widetilde{\{1, 2\}}$ 3. $\emptyset$ 4. D 5. $\{1, 2, 3, 4, 5\}$

6. $\{\tilde{1}\}$ 7. $\{7, 8, 9, 10\}$ 8. $\{7, 8, 9, 10\}$ 9. $\{2, 3, 4\}$

10. $\{3, 4\}$ 11. $\{2, 4, 6, 8, 10\}$ 12. $\{4, 6, 8, 10\}$

Answers for Quiz.

A.
1. $3a - 8$
2. $9x - 6$
3. $7a - 2$
4. 118
5. 1230
6. 4
7. n^2
8. $16 + 8x$
9. 1900
10. 1
11. x
12. 8/21
13. $21c/8$
14. $4b$
15. $2(d + 2)$

B.
1. $5(n - 2)$
2. $x(2 + 3y)$
3. $13r(2s - 1)$
4. $(b - 4)(b + 4)$
5. $4(c^2 - 15)$
6. $(d - 9)(d - 2)$
7. $(e - 9)(e + 2)$
8. $(f + 6)(f - 3)$
9. $(g + 6)(g + 3)$
10. $(h - 18)(h - 1)$
11. $(j + 18)(j - 1)$
12. $(k - 4)(k - 4)$
13. $(n - 8)(n + 2)$
14. $(4 + p)(4 + p)$
15. $(8 - q)(2 + q)$
16. $r(s - 3)(s - 3)$
17. $3(t - 4)(t - 4)$
18. $(\frac{1}{2}u - \frac{1}{3}x)(\frac{1}{2}u + \frac{1}{3}x)$

C.
1. 3
2. 3
3. $\{y: y > 3\}$
4. 3.5
5. -15
6. 1
7. $\{d: 1 < d\}$
8. 6, -6
9. 0, 1

Skill Quiz.

A. Simplify.

1. $3(a - 4) + 4$
2. $6(2x - 1) - 3x$
3. $5a - 2(1 - a)$
4. $(65 + 18) + 35$
5. $8 \times 123 + 2 \times 123$
6. $(4 + x^2) - x^2$
7. $(n - 5)(n + 5) + 25$
8. $(4 + x)^2 - x^2$
9. $75 \times 19 + 19 \times 25$
10. $\frac{2}{3} + \frac{3}{4} - \frac{5}{12}$
11. $\frac{2x}{3} - \frac{5x}{12} + \frac{3x}{4}$
12. $\frac{10}{27} \times \frac{36}{35}$
13. $\frac{35c^2}{36} \div \frac{10c}{27}$
14. $12ab \div (3a)$
15. $\frac{4(d + 2)^2}{2(d + 2)}$

B. Factor.

1. $5n - 10$
2. $2x + 3xy$
3. $26rs - 13r$
4. $b^2 - 16$
5. $4c^2 - 60$
6. $d^2 - 11d + 18$
7. $e^2 - 7e - 18$
8. $f^2 + 3f - 18$
9. $g^2 + 9g + 18$
10. $h^2 - 19h + 18$
11. $j^2 + 17j - 18$
12. $k^2 - 8k + 16$
13. $n^2 - 6n - 16$
14. $p^2 + 8p + 16$
15. $16 + 6q - q^2$
16. $rs^2 - 6rs + 9r$
17. $3t^2 - 24t + 48$
18. $\frac{1}{4}u^2 - \frac{1}{9}x^2$

C. Solve. [For the inequations, give the solution set, using the simplest sentence possible as set selector.]

1. $3x - 8 = 13 - 4x$
2. $10 - 2v = v + 1$
3. $2y + 7 > y + 10$
4. $\frac{r}{3} = \frac{7}{6}$
5. $\frac{s + 3}{4} = \frac{s}{5}$
6. $\frac{15}{4c} = \frac{3}{4c} + 3$
7. $14 - 3d < 2 + 9d$
8. $e^2 = 36$
9. $a(a - 1) = 0$

*

the relation of being-$^+2$-greater-than [the second component of each of its ordered pairs is $^+2$ greater than its first component].

[On terminology.--Many people, including some mathematicians, feel that a relation is something other than a set of ordered pairs [although all would probably agree that this something "gives" one such a set]. However, it seems to be very difficult for one who feels this way about relations to convey his meaning of 'relation' to someone who doesn't already feel the same way. Since the concept of a relation as being merely a set of ordered pairs is adequate for mathematics [and for much of logic], and since this concept involves nothing new to students who have already worked with sets and ordered pairs, we have chosen this simpler one of the two alternatives. Similar remarks apply to the notion of operation and to the yet-to-be-introduced notion of function. The definition of 'function' which we adopt in Section 5.05 is that a function is a set of ordered pairs no two of which have the same first component. So, an operation is a function, and a function is a special kind of relation. [Functions are sometimes called many-one relations.] There seems to be no explicit convention as to when one uses the word 'operation' in preference to 'function'.]

*

[Please see TC[5-10]d for an important note on Supplementary Exercises.]

*

Note that the Miscellaneous Exercises are really review exercises for Units 1, 2, 3, and 4. With a very few exceptions, they can be assigned independently of the work you are doing in Unit 5. We have included work on manipulation, solving of equations and inequations, word problems, business arithmetic, percent, conversion of units, and radicals. [The exercises on radicals are arranged in developmental fashion, and should be assigned just before the work on quadratic equations.] On TC[5-K]e we give a suggested Skill Quiz. Results on this quiz may provide you with an opportunity for making differentiated assignments from the Miscellaneous Exercises.

Students who have guessed DeMorgan's Laws from Exercises 3 and 4 in Part F may use them to obtain short cuts for answering parts (k) and (n) here. [Since $\widetilde{T \cap U} = \tilde{T} \cup \tilde{U}$, $n(\tilde{T} \cup \tilde{U}) = n(\widetilde{T \cap U})$. So, the answer for part (k) is the same as the answer for part (j).] Those who have not guessed DeMorgan's Laws may be led to do so by your asking whether it is merely a coincidence that parts (j) and (k) have the same answer, and that parts (m) and (n) have the same answer.

*

Answers for Part I [on page 5-K].

1. The experts' UPPER TRIANGLE contains 4950 ordered pairs $[(100^2 - 100)/2 = 4950]$.

   The ordered pairs named in (b), (c), (g), (j), and (ℓ) belong to the experts' UPPER TRIANGLE.

2. An ordered pair of the 100-by-100 lattice belongs to UPPER TRIANGLE if and only if its second component is Greater than its first component.

*

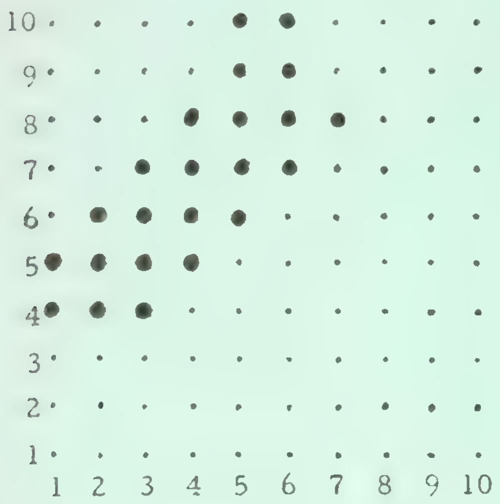
Exercise 2 of Part I is a first step toward associating a set of ordered pairs [in this case, UPPER TRIANGLE] with a relation [in this case, the relation $>$ among the positive integers from 1 through 100]. The purpose of section 5.01 is to accustom students to accepting relations as being sets of ordered pairs.

*

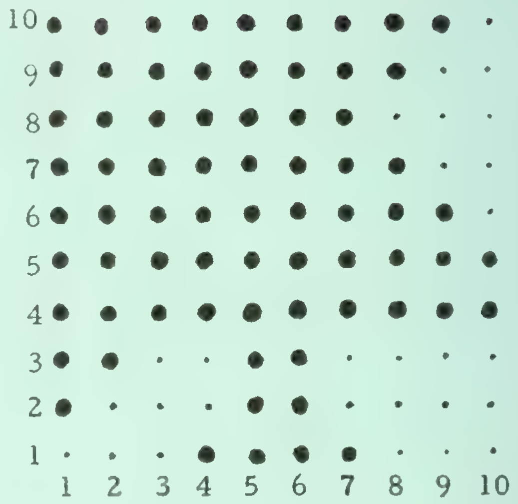
In Unit I students learned that an operation is a set of ordered pairs. [They will be reminded of this on page 5-79]. For example, that the operation adding $^{+}2$ is the set whose members are such ordered pairs of real numbers as $(^{-}3, ^{-}1)$, $(^{+}0.5, ^{+}2.5)$, and $(\sqrt{2}, 2 + \sqrt{2})$. On page 1-107, students were told that, more specifically, an operation is a set of ordered pairs no two of which have the same first component. So, an operation is a special kind of relation. Often, the name given to a set of ordered pairs which is an operation is different according to whether one is thinking of it from "the operation point of view" or from "the relation point of view". Thus, the operation adding $^{+}2$ is

Answers for Part H [on page 5-K].

1.



2.



3. (a) 50 (b) 50 (c) 45 (d) 55 (e) 100 (f) 0 (g) 100

(h) 0 (i) 23 (j) 77 (k) 77 (ℓ) 72 (m) 28 (n) 28

*

In doing parts (b), (d), (e), (g), and (j) of Exercise 3 students will probably become aware of and use the generalization: For each subset A of a space S, $n(A) + n(\widetilde{A}) = n(S)$. [As on page 4-12, '$n(A)$' means the number of elements in A.] They may also use this to get a short cut for solving part (ℓ) [$n(T \cup U) = 100 - n(\widetilde{T \cup U})$]. So, they may solve (m) first, and use the result to solve part (ℓ). On the other hand, they may answer part (ℓ) by recalling the generalization given on page TC[4-12]a and using the instance:

$$n(T \cup U) = n(T) + n(U) - n(T \cap U),$$

together with the results of parts (a), (c), and (i). Point out, if they do not use this to answer part (ℓ), that they may use it to check the consistency of their answers for parts (ℓ), (a), (c), and (i). The consistency of their answers for parts (k), (b), (d), and (n) can be checked in a similar manner.

Notice that Part H begins with the sentence:

> For each two games that the members of the Games Club invent, they can make up more games just by combining the charts for the two games.

Ask your students whether there could be pairs of games whose intersection will not give a new game. The intersection of TREE and UPPER TRIANGLE INTERSECTION TREE will not give a new game since $U \cap T \subseteq T$. Also, ask the students: If the intersection of two games does not give a new game, can the union of these two games give a new game? [If the set of points for one game is a subset of the set of points for another game, neither the intersection of these two sets nor their union will give a new game.]

*

Here is a description of a device you may find useful in illustrating the notions of union, intersection, and complement, and demonstrating principles such as DeMorgan's Laws. Draw a large-scale graph of T, say, on a sheet of typing paper. Then cut out 1/4-inch squares, each with its center at a graph of a member of T, and with its sides parallel to the edges of the paper. [This gives you a sheet with fifty square holes cut in it.] Indicate the graphs of the members of $\widetilde{T}$ by moderately heavy dots. Label the sheet 'T'. Now, do the same thing for U, for $\widetilde{T}$, and for $\widetilde{U}$, making sure that, when you place one sheet on another, graphs of the same ordered pair coincide.

Now, if you place, say, the T-sheet on the U-sheet, members of $T \cap U$ are identified by holes, and members of $T \cup U$ by holes or thin spots. It is clear, for example, that $T \cap U$ is a subset of both T and U, and that both T and U are subsets of $T \cup U$. Using the T-sheet and the $\widetilde{T}$-sheet shows that $T \cup \widetilde{T} = D \times D$ and that $T \cap \widetilde{T} = \emptyset$. That $\widetilde{T \cap U} = \widetilde{T} \cup \widetilde{U}$ can be seen by comparing what one sees in placing the T-sheet over the U-sheet with what one sees on placing the $\widetilde{T}$-sheet over the $\widetilde{U}$-sheet. In the same way, one can see that $\widetilde{T \cup U} = \widetilde{T} \cap \widetilde{U}$. Other uses of these sheets, perhaps supplemented by sheets for additional relations, may occur to you as you use them.

*

In the case of genetic relationships, our convention leads, for example, to identifying fatherhood with $\{(x, y) \in P \times P: \ y \text{ is the father of } x\}$. [Here, 'P' denotes the set of people.] So, 'John is the father of Henry' is equivalent to '(Henry, John) $\in$ Fatherhood. Unfortunately this conflicts with the common usage of logicians who commonly make 'John is the father of Henry' equivalent to '(John, Henry) $\in$ Fatherhood. [Their convention is that a relation R is $\{(x, y): \ x \text{ bears the relation } R \text{ to } y\}$.] Although this divergence in usage is unfortunate, we believe it best to remain consistent with mathematical usage as it has been established in the important case of functional relations.

To avoid possible later confusion, note that when we speak of a relation among the members of a set, we do not mean to imply that all members of the set "get into the act". For example, cousinhood is a relation among people even though not everyone has a cousin. In this connection, you may want to glance over the introductory paragraph in section 5.04.

*

The objective to be reached by studying the material on pages 5-1 through 5-4 is student acceptance of the facts that one can come to know a relation by studying a certain set of ordered pairs and that it is natural to say that the relation is this set of ordered pairs. See TC[5-K]c, d.

*

One who identifies relations with sets of ordered pairs must decide whether a given relation R is $\{(x, y): y$ bears the relation R to $x\}$ or $\{(x, y): x$ bears the relation R to $y\}$. If he asks the question 'What relation does a nonnegative number bear to its square?', he gets as answer 'The square rooting relation.'. [A nonnegative number is the square root of its square.] Now, the square rooting relation is the function $\sqrt{\ }$ and, in graphing this function, the convention is to draw a picture of $\{(x, y), x \geq 0: y = \sqrt{x}\}$. So, by this convention, the square rooting relation is $\{(x, y), x \geq 0: y$ bears the square rooting relation to $x\}$. Since it would be quite unrealistic to consider changing this basic mathematical convention, and since we feel it to be very important to maintain consistency, we choose to identify each relation R with $\{(x, y): y$ bears the relation R to $x\}$. Consequently, in the Introduction, '5 TREES 3', for example, is equivalent to '$(3, 5) \in T$'. One result of this is that the relation of being greater than which each number bears to each smaller number is $\{(x, y): y > x\}$. So, for each x, for each y, $(x, y) \in >$ if and only if $y > x$. Similarly, the relation of being a factor of, with respect to the set I^+ of positive integers is $\{(x, y) \in I^+ \times I^+: y$ is a factor of x with respect to $I^+\}$. This relation is usually denoted by '$|$', so, $y \mid x$--that is, y is a factor of x with respect to I^+, if and only if $(x, y) \in |$.

Correction. In part (a) of Exercise 2 on page 5-7, change 'M' to 'B'.

In preparation for Exercise 1(f) of Part A, you may need to remind your students that, for each $x > 0$, $\sqrt{x}$ is the positive number whose square is x, and $-\sqrt{x}$ is the negative number whose square is x. Also, for Exercise 5, students may need to be reminded of the distinction between rational and irrational numbers [page 4-43 et. seq.] and of the fact [page 4-48] that, for each positive integer n, if $\sqrt{n}$ is not an integer, then $\sqrt{n}$ is irrational.

*

Answers for Part A [on pages 5-7, 5-8, and 5-9].

1. (a) T (b) T (c) T (d) F (e) F (f) T

   [Note the interesting connections in Exercise 1 between (a) and (c), and among (a), (d), and (e).]

2. (a) T (b) T (c) F [$7.8 \notin I$] (d) F (e) T (f) F

3. (3, 0), (8, −3), (−2, 3) [For each integer k, $(3 + 5k, -3k) \in A$.]

4. [Since 3 is a factor of 3 and of 6 with respect to I but is not a factor of 7, $D = \emptyset$. So, the instructions cannot be carried out.]

*

The set selectors in Exercises 3 and 4 [for which the domain of 'x' and 'y' is the set of integers] are examples of Diophantine equations. [See TC[4-41, 42]b and, for problems leading to such equations, TC[3-81] and TC[4-106, 107, 108]b and c.] The facts concerning the existence of solutions of such equations can be stated as follows. For integers a, b, and c, there are integers x and y such that $ax + by = c$ if and only if each common factor of a and b is a factor of c. It is easy to see that there are no solutions if this condition is not satisfied. It is more difficult to see that there are solutions whenever the condition is satisfied.

The bracketed remark made above in connection with the answer for Exercise 3 suggests how you can obtain all solutions of such an equation when one has been found. [In the case of Exercise 3, the components of a solution "in integers" other than (3, 0) must differ from 3 and 0 by integers, and if the sum of 3 times the first component and 5 times the second component is still to be 9, then the sum of 3 times the first difference and 5 times the second difference must be 0.] An interesting question to ask now in connection with Exercise 3 is 'Does A contain any point in the first quadrant?'. The answer is 'no', because there is no integer k such that $3 + 5k > 0$ and $-3k > 0$.

5. $(\sqrt{2}, \sqrt{5})$, $(-\sqrt{2}, \sqrt{5})$, $(\sqrt{2}, -\sqrt{5})$

6. (a) T (b) T (c) T (d) F (e) F (f) T

7. [Since there are many correct answers for each of most of the parts of Exercise 7, there is little point in giving answers here. The usual procedure of substituting for one variable in the set selector and solving the resulting sentence will work for most parts.

   (j) Each solution in (r, s) of '$r = 2s$' belongs to the relation and, also, each solution in (r, s) of '$r^3 + 3s^2 = 11$' belongs to the relation.

   (k) The only correct answer is: (1, 1)

   (ℓ) (4, −2), (8, −5), (12, −8), ... all belong to the relation. In fact, the relation is $\{(x, y):$ for some $z \in I^+$, $x = 4z$ and $y = 1 - 3z\}$.

   (m) The relation is $\{(x, y):$ for some $k \in I$, $x = 3k$ and $y = (7 - 3k)/2\}$. So, some of the ordered pairs in the relation are (−3, 5), (0, 7/2), and (3, 2).

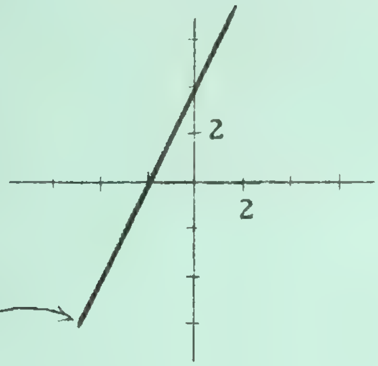
   (n) The relation is $\{(x, y):$ for some $k \in I$, $x = 15(2k + 1)$ and $y = 30(1 - k)\}$. So, some of the ordered pairs in the relation are (−15, 60), (15, 30), (45, 0), and (75, −30).

   (o) Here is an easy kind of answer: (George Washington, Mrs. George Washington).

   (p) Perhaps the baseball coach will help you grade this question.]

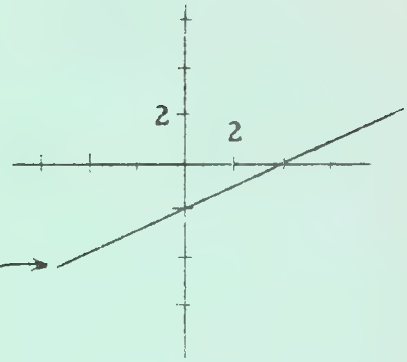
Answers for Part B.

1.



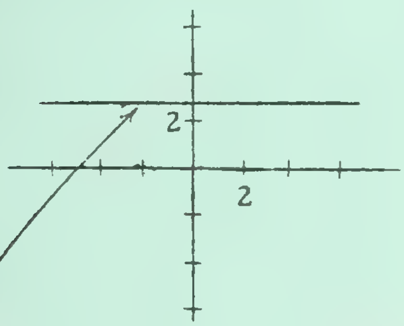
$\{(x, y): \ y = 2x + 4\}$

2.



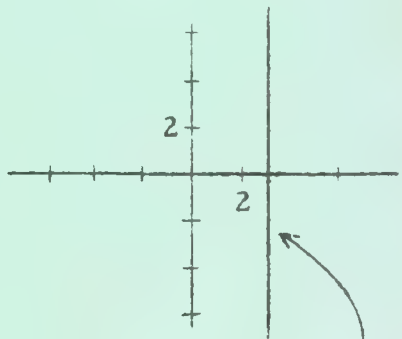
$\{(x, y): \ x = 2y + 4\}$

3.



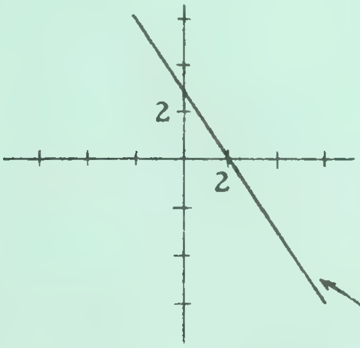
$\{(x, y): \ y = 3\}$

4.



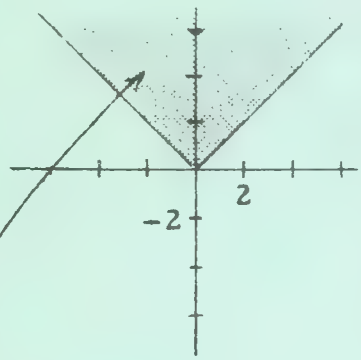
$\{(x, y): \ x = 3\}$

5.



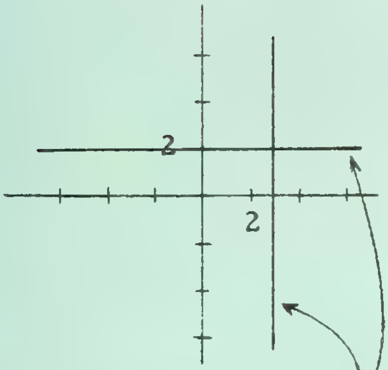
$\{(a, b): \ 3a + 2b - 6 = 0\}$

6.



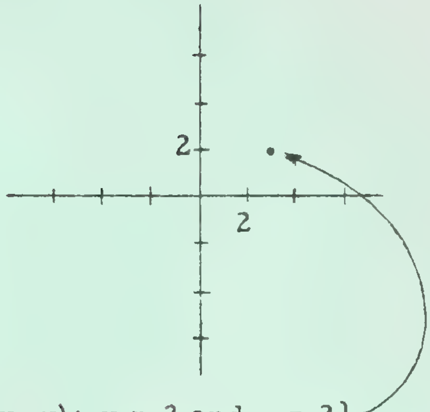
$\{(r, s): \ s \geq |r|\}$

7.



$\{(x, y):\ x = 3 \text{ or } y = 2\}$

8.



$\{(x, y):\ x = 3 \text{ and } y = 2\}$

*

Your students will probably notice

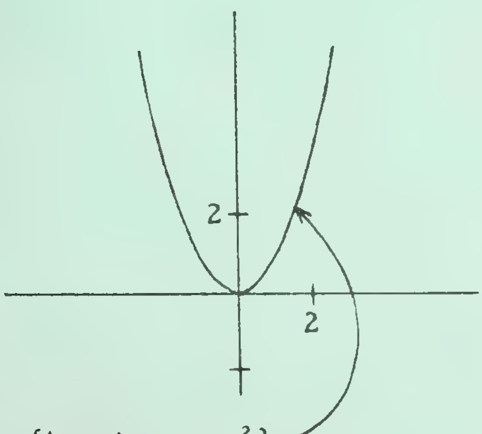
that $\{(x, y):\ x = 3 \text{ or } y = 2\} = \{(x, y):\ x = 3\} \cup \{(x, y):\ y = 2\}$

and $\{(x, y):\ x = 3 \text{ and } y = 2\} = \{(x, y):\ x = 3\} \cap \{(x, y):\ y = 2\}$

Finding other names using '$\cup$' or '$\cap$' for the relations given in Exercises 11-17 will provide a good review of the connections between 'and' and 'intersection', and between 'or' and 'union'. It may also make the graphing a little easier. See TC[5-H]a.

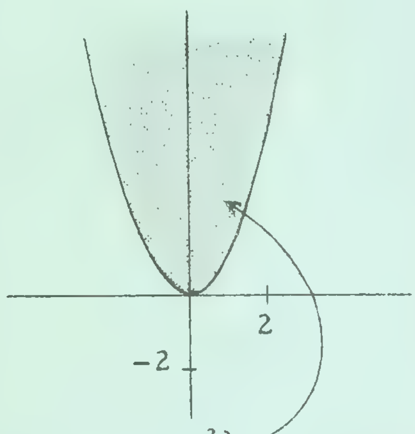
*

9.



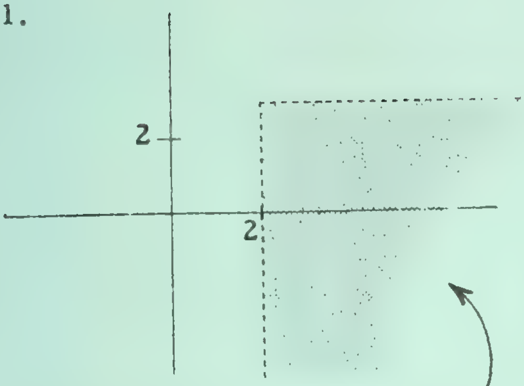
$\{(x, y):\ y = x^2\}$

10.



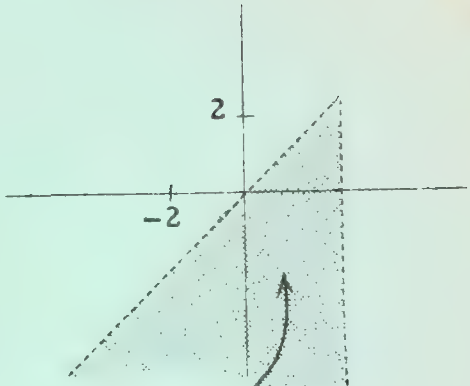
$\{(x, y):\ y \geq x^2\}$

11.



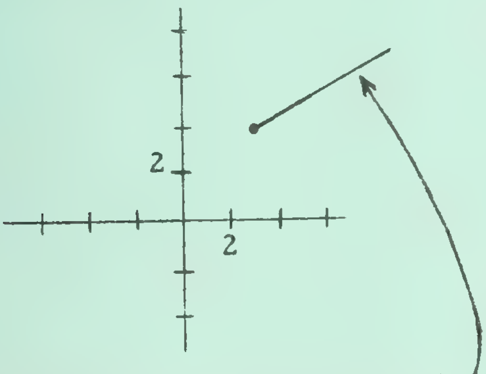
$\{(x, y):\ x > 2 \text{ and } y < 3\}$

12.



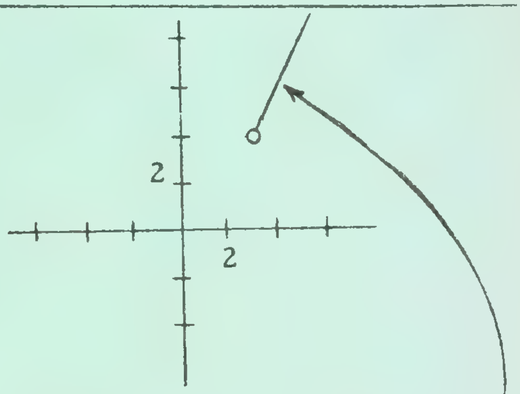
$\{(x, y):\ y < x < 3\}$

13.



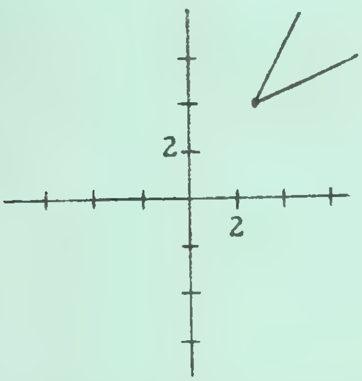
$\{(x, y):\ 2y = x + 5 \text{ and } x \geq 3\}$
[This relation is a ray.]

14.



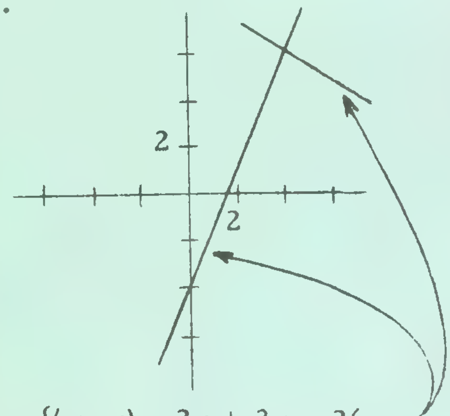
$\{(x, y):\ y = 2x - 2 \text{ and } y > 4\}$
[This relation is a half-line.]

15.



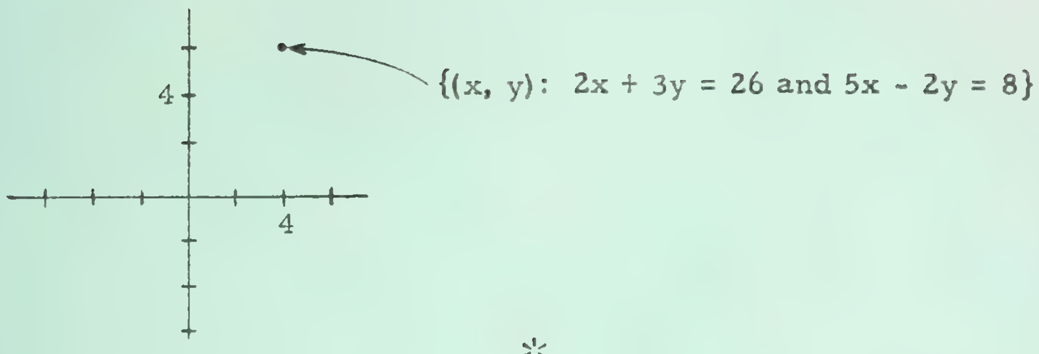
$\{(x, y):\ (2y = x + 5 \text{ and } x \geq 3) \text{ or } (y = 2x - 2 \text{ and } y \geq 4)\}$
[This relation is an angle.]

16.



$\{(x, y):\ 2x + 3y = 26 \text{ or } 5x - 2y = 8\}$

17.



*

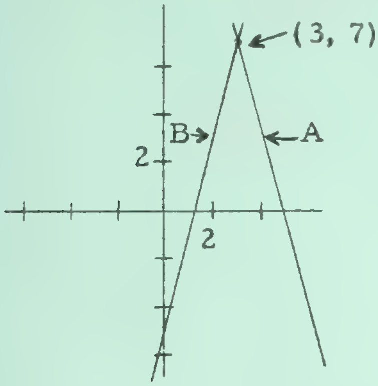
Note the bracketed sentences following the last exercise in Part B. Our practice with regard to Supplementary Exercises in Units 1-4 was to provide you with exercises for students who needed more drill. In Unit 5 we are modifying this practice to include in the Supplementary Exercises not only more drill work but also material for the student who wants harder problems or who has the time and interest to pursue additional topics. We strongly urge that you acquaint yourself with Supplementary Exercises well in advance of the time you may want to assign them. Then you will be prepared to make supplementary [and differentiated] assignments on a moment's notice and you will avoid the unfortunate situation of assigning impossibly difficult problems to slower students.

*

The comparison made [in the lower third of page 5-10] between Exercises 16 and 17 of Part B can also be made between Exercises 7 and 8 of Part B. Students should be asked to point out this second pair of exercises.

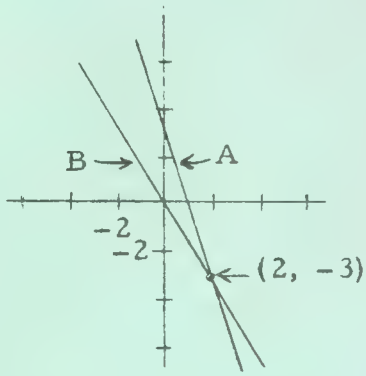
Answers for Part A [which begins on page 5-11].

1.



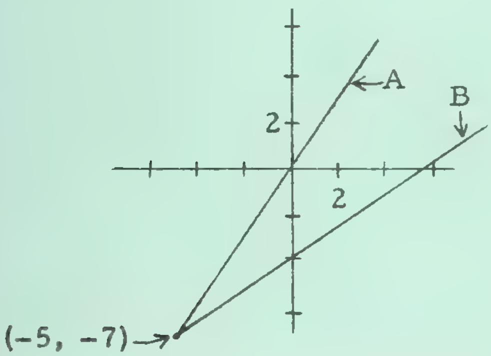
$A \cap B = \{(3, 7)\}$

2.



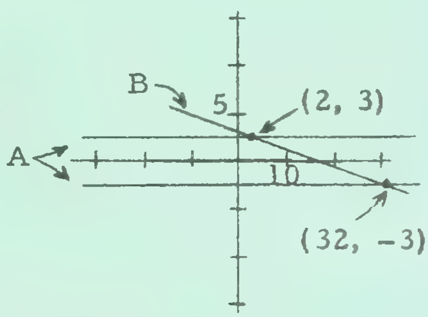
$A \cap B = \{(2, -3)\}$

3.



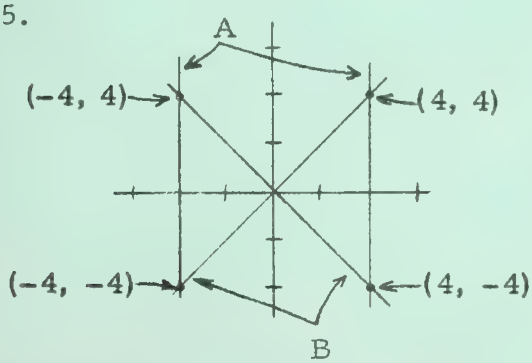
$A \cap B = \{(-5, -7)\}$

4.



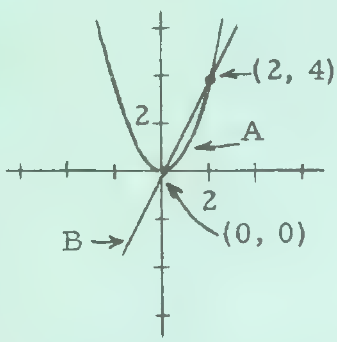
$A \cap B = \{(2, 3), (32, -3)\}$

5.



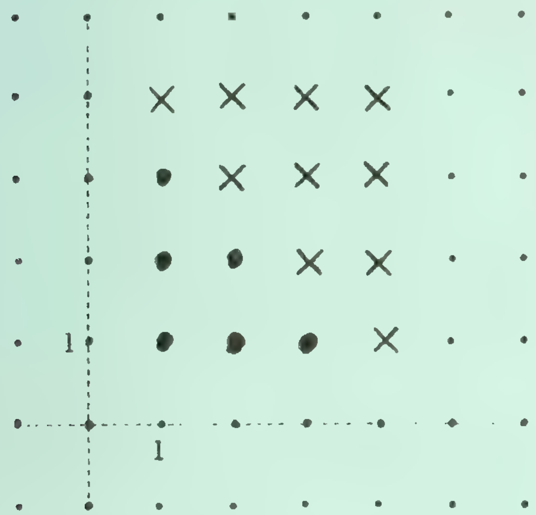
$A \cap B = \{(4, 4), (4, -4), (-4, 4), (-4, -4)\}$

6.



$A \cap B = \{(0, 0), (2, 4)\}$

7. There are 16 ordered pairs in $A \cup B$. Students can answer this question by drawing the graphs or by listing the pairs. You may also want to accept '$\{(x, y) \in I \times I\colon\ 0 < x < 5 \text{ and } 0 < y < 5\}$' as an answer.

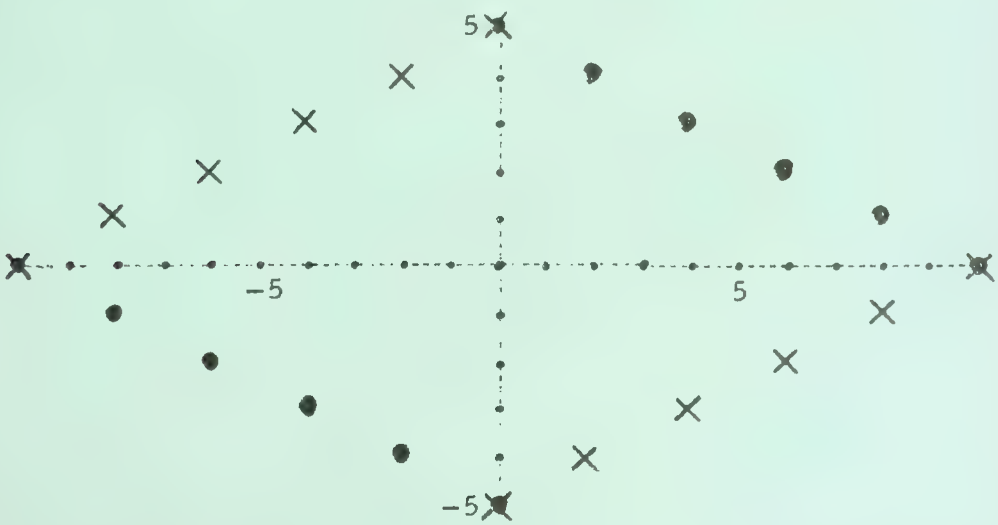


The graph of A consists of dots.

The graph of B consists of crosses.

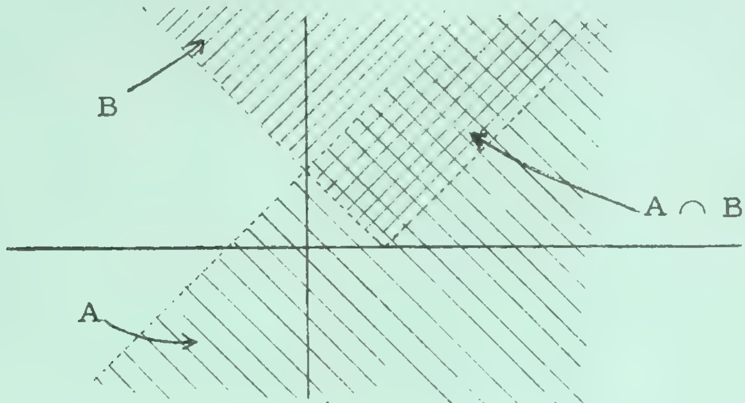
The ordered pairs in $A \cup B$ are those whose graphs are marked by either a dot or a cross [or both].

8. The graph of A consists of dots, the graph of B consists of crosses, and the ordered pairs in $A \cup B$ are those whose graphs are marked by either a dot or a cross [or both].



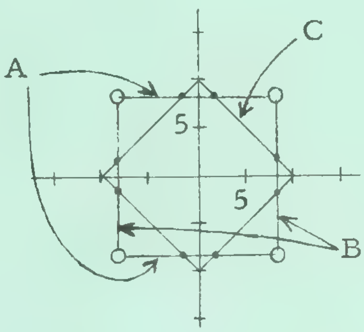
Answer for Part B [on page 5-12].

[This exercise is preparation for the geometric problem in section 5.03.]



*

Answers for Part C [on page 5-12].



2. $A \cup B$ is a square with the corners missing.

   $(A \cup B) \cap C$

   $= \{(8, 2), (2, 8), (-2, 8), (-8, 2), (-8, -2), (-2, -8), (2, -8), (8, -2)\}$

3. $A \cap C = \{(2, 8), (-2, 8), (-2, -8), (2, -8)\}$

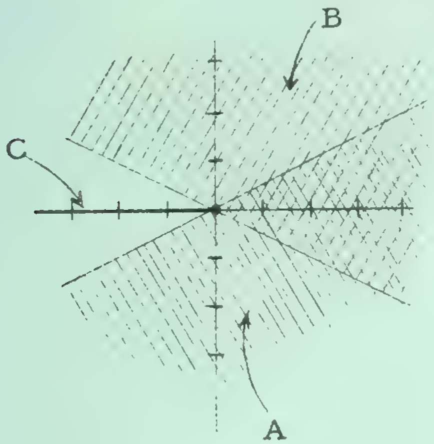
   $B \cap C = \{(8, -2), (8, 2), (-8, 2), (-8, -2)\}$

   So, $(A \cap C) \cup (B \cap C) = (A \cup B) \cap C$.

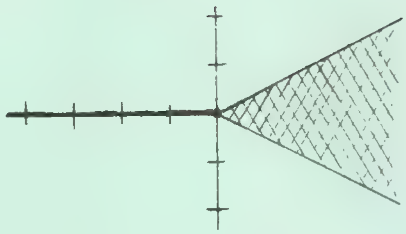
*

Answers for Part D [on page 5-12].

1.



2.



3. $(A \cup C) \cap (B \cup C) = (A \cap B) \cup C$

*

Part C suggests that the operation intersecting is distributive with respect to the operation unioning.

Part D suggests that the operation unioning is distributive with respect to the operation intersecting.

These suggestions are developed in section 5.02 which begins on page 5-13.

Quiz

A. True or false?

1. $(3, -8) \in \{(x, y): y < 3x - 15\}$ 2. $(2, -1) \in \{(x, y): x = y^2 + 1\}$

3. $(5, 1) \in \{(r, s): s + 7r - 41 = 1 + 4s\}$

4. $(8, 0) \in \{(u, v), u \geq 0: v < \sqrt{u} - 3\}$

5. $(17, 3) \notin \{(a, b): a + 3 < 5b + 4\}$

B. Draw graphs and find the ordered pairs in $A \cap B$.

1. $A = \{(x, y): 4x - 3 = 2y\}$, $B = \{(x, y): 6x + 3y = -\frac{9}{2}\}$

2. $A = \{(r, s): |s| = 3\}$, $B = \{(r, s): 3s = r\}$

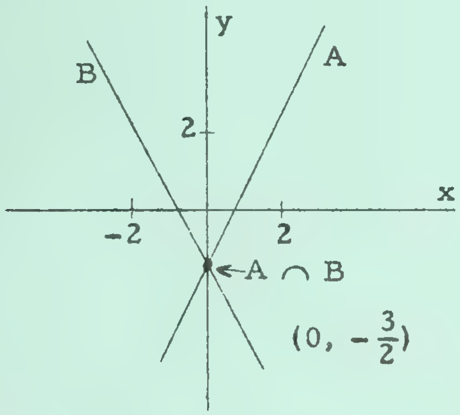
3. $A = \{(x, y) \in I \times I: x + y = 7\}$, $B = \{(x, y) \in I \times I: -3 \leq y \leq 3\}$

*

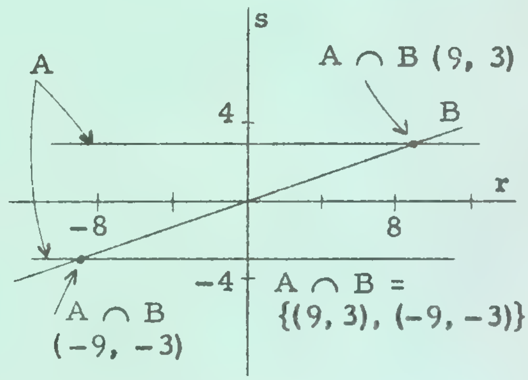
Answers for Quiz.

A. 1. T 2. T 3. F 4. F 5. T

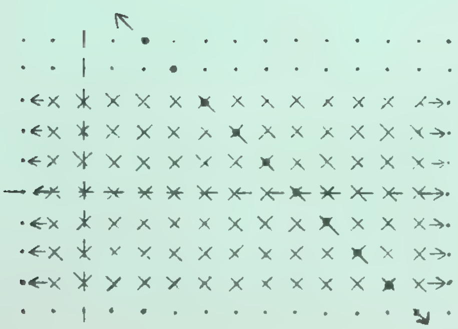
B. 1.



2.



3.



The graph of A consists of heavy dots.

The graph of B consists of crosses.

The ordered pairs in $A \cap B$ are those whose graphs are marked by a dot <u>and</u> a cross. These pairs are $(10, -3)$, $(9, -2)$, $(8, -1)$, $(7, 0)$, $(6, 1)$, $(5, 2)$, and $(4, 3)$.

Section 5.02 has two purposes:

(1) to remind students of some principles which they have already discovered for operating on sets and to acquaint them with other such principles, and

(2) to point out analogies between operations on sets and operations on numbers, and show students that theorems about sets can be derived from basic principles in much the same way that, in Unit 2, they derived theorems about numbers from basic principles.

[A list of basic principles and theorems about sets is given in the SUMMARY on pages 5-22 and 5-23.]

To save class time, exercises on proving theorems have, for the most part, been made optional [see Note at foot of page 5-18].

*

There are logical difficulties involved in the notion of "the set of all sets", and besides, the discussion of complementing requires specification of some containing set or space [see TC[5-F]b]. Hence, in discussing (2) on page 5-13, we specify that the domain of 'x', 'y', and 'z' should consist of all subsets of some set S, and refer to (2) as the dpiu <u>for subsets of S</u>.

*

Part D on page 5-12 [referred to on page 5-14] suggests the distributive principle for unioning over intersecting, for subsets of the number plane:

$$\forall_x \forall_y \forall_z \ (x \cap y) \cup z = (x \cup z) \cap (y \cup z)$$

This principle continues to hold when the domain of 'x', 'y', and 'z' is taken to be the set of subsets of any set S. In that case, it is <u>the distributive principle for unioning over intersecting, for subsets of S</u>.

*

Answers for Part A.

[See page 5-16 for statements of the commutative and associative principles for unioning and intersecting.]

*

Answers for Part B.

$\forall_x \; x \cup \emptyset = x; \; \forall_x \; x \cap \emptyset = \emptyset$

*

Answers for Part C [on page 5-15].

$\forall_x \; x \cup x = x; \; \forall_x \; x \cap x = x$

For a discussion of the kind of testing-pattern shown at the top of page 5-17, see TC[2-64]a, b.

*

Answers for Part D [on pages 5-17 and 5-18].

1 [We shall not rewrite the test-pattern here. In doing so, be sure to make the indicated replacements in <u>both</u> <u>columns</u> of the pattern.] Yes

2. After rewriting one has the same statements he started with. [The fact that they are listed in a different order is unimportant.]

3 By making the replacements specified in Exercise 1.

*

A sentence like '$\forall_x \forall_y\ (x \cup y) \cap (\tilde{x} \cap \tilde{y}) = \emptyset$' contains only two kinds of symbols--logical symbols [that is, symbols such as '$\forall$', 'x', '$=$', 'if... then---', etc.] and symbols relating to sets [that is, symbols like '$\cup$', '$\cap$', '$\sim$', '$\emptyset$', and 'S']. Such a sentence is transformed into its dual by the replacements specified in Exercise 2 of Part D. For example, the dual of the sentence mentioned above is '$\forall_x \forall_y\ (x \cap y) \cup (\tilde{x} \cup \tilde{y}) = S$'. In doing the exercises of Part D the student should discover that [since dualizing does not modify the grammatical structure of a sentence] the result of dualizing the sentences in a test-pattern is again a test-pattern; and that, since the dual of each basic principle is again a basic principle, it follows that the dual of each theorem is a theorem. This point should be discovered by the time the students have completed Exercise 3. Exercise 4 on page 5-18 gives an opportunity for checking the validity of this discovery. [As will be seen in the COMMENTARY for page 5-21, in dualizing a sentence which contains '$\subseteq$' one must, in addition to making the replacements specified in Exercise 2 of Part D, replace '$\subseteq$' by '$\supseteq$'.]

*

Replacement Table for Dualizing

$\cap$	$\cup$	$\emptyset$	S	$\subseteq$	$\supseteq$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\cup$	$\cap$	S	$\emptyset$	$\supseteq$	$\subseteq$

4. [The test-pattern asked for is obtained by dualizing the one given in the exercise. It begins with '$x = x \cap S$ [$\forall_x\ x \cap S = x$]', and ends with '$= x \cap x$[$\forall_x\ x \cup \emptyset = x$]'.]

*

Answer for Part E.

Suppose that $x \cup y = \emptyset$.

Then

$$
\begin{aligned}
y &= y \cup (x \cup y) && [\forall_x\ x \cup \emptyset = x] \\
&= (x \cup y) \cup y && [\forall_x \forall_y\ x \cup y = y \cup x] \\
&= x \cup (y \cup y) && [\forall_x \forall_y \forall_z\ (x \cup y) \cup z = x \cup (y \cup z)] \\
&= x \cup y && [\forall_x\ x = x \cup x] \\
&= \emptyset. && [\text{Assumption: } x \cup y = \emptyset]
\end{aligned}
$$

So, if $x \cup y = \emptyset$ then $y = \emptyset$.

*

Notice that the theorem proved in Part E shows that the analogy between adding and unioning, multiplying and intersecting, 0 and $\emptyset$, and 1 and S cannot be extended to include an operation on subsets of S analogous to the operation oppositing on real numbers. For if there were such an operation *, then the analogue, '$\forall_x\ x \cup * x = \emptyset$' of the principle of opposites would have to be a theorem about subsets of S. But, if so, by the theorem proved in Part E, '$\forall_x * x = \emptyset$' would be a theorem. And this last is analogous to the statement '$\forall_x\ -x = 0$' about real numbers. Since <u>this</u> statement is not a theorem about real numbers, the only candidate for an analogue of oppositing [the operation on sets which maps each set on $\emptyset$] is <u>not</u> such an analogue. Hence, oppositing has no analogue in the algebra of sets.

Skill Quiz.

A. Simplify.

1. $(3a + 2)(3a - 2) + (2a - 3)(3a + 1)$
2. $57a^2bc^3 \div (3abc^2)$
3. $7x - 3y - (-4y + 8x)$
4. $\frac{7n}{5} + \frac{n}{3}$
5. $\frac{(c + 2)^2}{33} \cdot \frac{44}{4c + 8}$
6. $51e^5(f + g)^2 \div [17e^6 (f + g)]$
7. $3(r - s)^2 - 3(s^2 + 2rs + r^2)$
8. $\frac{9 - d}{54} + \frac{d - 6}{36}$

B. Factor completely.

1. $49 - y^2$
2. $a^2 - a - 12$
3. $5de^2 - 5d$
4. $16f^2 - 40f + 25$
5. $12n^2 - nr - 6r^2$
6. $12t^2 + 89jt - 56j^2$

C. Solve. [For the inequations, give the solution set, using the simplest sentence possible as set selector.]

1. $3q^2 = 108$
2. $5s - 12 = 9s + 6$
3. $3z + 2 > 11$
4. $\frac{7}{12} = \frac{k}{4}$
5. $(2 - a)(3 + a) = \frac{5a - 15}{-2.5}$
6. $5p + 7 < 7p - 5$

*

Answers for Quiz.

A. 1. $15a^2 - 7a - 7$
2. $19ac$
3. $-x + y$
4. $\frac{26n}{15}$
5. $\frac{c + 2}{3}$
6. $\frac{3(f + g)}{e}$
7. $-12rs$
8. $\frac{d}{108}$

B. 1. $(7 - y)(7 + y)$
2. $(a - 4)(a + 3)$
3. $5d(e - 1)(e + 1)$
4. $(4f - 5)^2$
5. $(4n - 3r)(3n + 2r)$
6. $(12t - 7j)(t + 8j)$

C. 1. $6, -6$
2. -4.5
3. $\{z: z > 3\}$
4. $7/3$
5. $0, 1$
6. $\{p: 6 < p\}$

and either of the principles for complements from the other. For example:

$$\tilde{x} \cup \emptyset = \tilde{x} \qquad [\forall_x\ x \cup \emptyset = x]$$

$$\tilde{x} \cup \tilde{S} = \tilde{x} \qquad [\tilde{S} = \emptyset]$$

$$\widetilde{x \cap S} = \tilde{x} \qquad [\text{DeMorgan's Laws}]$$

$$\widetilde{\widetilde{x \cap S}} = \tilde{\tilde{x}}$$

$$x \cap S = x \qquad [\forall_x\ \tilde{\tilde{x}} = x]$$

and:

$$\tilde{\tilde{x}} \cup \tilde{x} = S \qquad [\forall_x\ \tilde{x} \cup x = S]$$

$$\tilde{\tilde{x}} \cup \tilde{x} = \tilde{\emptyset} \qquad [\tilde{\emptyset} = S]$$

$$\widetilde{\tilde{x} \cap x} = \tilde{\emptyset} \qquad [\text{DeMorgan's Laws}]$$

$$\widetilde{\widetilde{\tilde{x} \cap x}} = \tilde{\tilde{\emptyset}}$$

$$\tilde{x} \cap x = \emptyset \qquad [\forall_x\ \tilde{\tilde{x}} = x]$$

Finally, either of DeMorgan's Laws, and either of '$\tilde{S} = \emptyset$' and '$\tilde{\emptyset} = S$' can be derived from the other by using '$\forall_x\ \tilde{\tilde{x}} = x$'. For example:

$$\tilde{\tilde{x}} \cup \tilde{\tilde{y}} = \widetilde{\tilde{x} \cap \tilde{y}} \qquad [\forall_x \forall_y\ \widetilde{x \cap y} = \tilde{x} \cup \tilde{y}]$$

$$x \cup y = \widetilde{\tilde{x} \cap \tilde{y}} \qquad [\forall_x\ \tilde{\tilde{x}} = x]$$

$$\widetilde{x \cup y} = \widetilde{\widetilde{\tilde{x} \cap \tilde{y}}}$$

$$\widetilde{x \cup y} = \tilde{x} \cap \tilde{y} \qquad [\forall_x\ \tilde{\tilde{x}} = x]$$

and:

$$S = \tilde{\emptyset}$$

$$\tilde{S} = \tilde{\tilde{\emptyset}}$$

$$\tilde{S} = \emptyset \qquad [\forall_x\ \tilde{\tilde{x}} = x]$$

So, we could take as basic principles, for example, the first of each of the pairs given on page 5-17 together with '$\forall_x \forall_y\ \widetilde{x \cap y} = \tilde{x} \cup \tilde{y}$', '$\forall_x\ \tilde{\tilde{x}} = x$', and '$S = \tilde{\emptyset}$'.

$$\emptyset \cup S = S \qquad [\forall_x\ \emptyset \cup x = x]$$

$$\emptyset \cap S = \emptyset \qquad [\forall_x\ x \cap S = x]$$

So, $\emptyset \cup S = S$ and $\emptyset \cap S = \emptyset$.

But, if $\emptyset \cup S = S$ and $\emptyset \cap S = \emptyset$ then $\widetilde{\emptyset} = S$. [the complement theorem]

Hence, $\widetilde{\emptyset} = S$.

$$\tilde{x} \cup x = S \qquad [\forall_x\ \tilde{x} \cup x = S]$$

$$\tilde{x} \cap x = \emptyset \qquad [\forall_x\ \tilde{x} \cap x = \emptyset]$$

So, $\tilde{x} \cup x = S$ and $\tilde{x} \cap x = \emptyset$

But, if $\tilde{x} \cup x = S$ and $\tilde{x} \cap x = \emptyset$ then $\tilde{\tilde{x}} = x$. [the complement theorem]

Hence, $\tilde{\tilde{x}} = x$.

*

DeMorgan's Laws cast more light on the duality discovered in solving Part D on pages 5-17 and 5-18. For example, using these laws and the principle '$\forall_x\ \tilde{\tilde{x}} = x$', one can derive the second distributive principle from the first. Here is a test-pattern.

$$(\tilde{x} \cup \tilde{y}) \cap \tilde{z} = (\tilde{x} \cap \tilde{z}) \cup (\tilde{y} \cap \tilde{z}) \qquad [\forall_x \forall_y \forall_z\ (x \cup y) \cap z = (x \cap z) \cup (y \cap z)]$$

$$\widetilde{(x \cap y)} \cap \tilde{z} = \widetilde{(x \cup z)} \cup \widetilde{(y \cup z)} \qquad [\text{DeMorgan's Laws}]$$

$$\widetilde{(x \cap y) \cup z} = \widetilde{(x \cup z) \cap (y \cup z)} \qquad [\text{DeMorgan's Laws}]$$

$$\widetilde{\widetilde{(x \cap y) \cup z}} = \widetilde{\widetilde{(x \cup z) \cap (y \cup z)}}$$

$$(x \cap y) \cup z = (x \cup z) \cap (y \cup z) \qquad [\forall_x\ \tilde{\tilde{x}} = x]$$

[The fourth line derives from the third by way of the logical principle '$\forall_x \forall_y$ if $x = y$ then $\tilde{x} = \tilde{y}$.' Cf. the corresponding uniqueness principles for intersecting, for which a test-pattern is given on page 5-17.]

In entirely similar manners one can derive either of the commutative and associative principles from the other. Moreover, if we use, in addition to DeMorgan's Laws and '$\forall_x\ \tilde{\tilde{x}} = x$', the principles '$\tilde{S} = \emptyset$' and '$\widetilde{\emptyset} = S$', we can derive either of the principles of $\emptyset$ and S from the other,

Below is a test-pattern for the complement theorem.

Suppose that $x \cup y = S$ and $x \cap y = \emptyset$.

It follows that $\tilde{x} \cap (x \cup y) = \tilde{x} \cap S$,

so, $(\tilde{x} \cap x) \cup (\tilde{x} \cap y) = \tilde{x} \cap S$, [$\ell$dpiu]

and $\emptyset \cup (\tilde{x} \cap y) = \tilde{x} \cap S$. [$\forall_x\ \tilde{x} \cap x = \emptyset$]

Hence, $\tilde{x} \cap y = \tilde{x}$. [$\forall_x\ \emptyset \cup x = x$; $\forall_x\ x \cap S = x$]

But, since $x \cap y = \emptyset$,

$(\tilde{x} \cap y) \cup (x \cap y) = \tilde{x} \cup \emptyset$,

so, $(\tilde{x} \cup x) \cap y = \tilde{x} \cup \emptyset$, [dpiu]

and $S \cap y = \tilde{x} \cup \emptyset$. [$\forall_x\ \tilde{x} \cup x = S$]

Hence, $y = \tilde{x}$, [$\forall_x\ S \cap x = x$; $\forall_x\ x \cup \emptyset = x$]

and $\tilde{x} = y$.

Consequently, if $x \cup y = S$ and $x \cap y = \emptyset$ then $\tilde{x} = y$.

[As above, we shall continue to use (without bothering to derive them) theorems, such as '$\forall_x\ \emptyset \cup x = x$', which can be derived from basic principles or previously proved theorems by using the commutative principles.]

*

Here are test-patterns for '$\tilde{S} = \emptyset$', '$\tilde{\emptyset} = S$', and '$\forall_x\ \tilde{\tilde{x}} = x$'.

$S \cup \emptyset = S$ [$\forall_x\ x \cup \emptyset = x$]

$S \cap \emptyset = \emptyset$ [$\forall_x\ S \cap x = x$]

So, $S \cup \emptyset = S$ and $S \cap \emptyset = \emptyset$.

But, if $S \cup \emptyset = S$ and $S \cap \emptyset = \emptyset$ then $\tilde{S} = \emptyset$. [the complement theorem]

Hence, $\tilde{S} = \emptyset$.

In order to save space horizontally, we have, in giving the first part of the test pattern for the first of DeMorgan's Laws, reverted to the form first introduced on page 2-35.

*

Answers for Part A.

1. [Such a test-pattern is given on TC[5-19]b.]

2. [The test-pattern asked for can be obtained merely by dualizing the test-pattern given on page 5-20 for the first of DeMorgan's Laws. The first lines obtained will be:

$$\left.\begin{array}{l} (x \cap y) \cup (\tilde{x} \cup \tilde{y}) \\ = [x \cup (\tilde{x} \cup \tilde{y})] \cap [y \cup (\tilde{x} \cup \tilde{y})] \end{array}\right\} \forall_x \forall_y \forall_z\ (x \cap y) \cup z = (x \cup z) \cap (y \cup z)$$

and it will end with:

So, by the complement theorem, $\widetilde{x \cap y} = \tilde{x} \cup \tilde{y}$.]

*

The complement theorem is a special case of <u>the cancellation theorem</u>:

$$\forall_x \forall_y \forall_z \text{ if } y \cup z = x \cup z \text{ and } y \cap z = x \cap z \text{ then } x = y$$

[More precisely, the complement theorem is an easy consequence of the cancellation theorem, the principles for complements, and the commutative principles. For, an immediate consequence of the cancellation theorem is:

$$\forall_y \forall_z \text{ if } y \cup z = \tilde{z} \cup z \text{ and } y \cap z = \tilde{z} \cap z \text{ then } \tilde{z} = y]$$

Below is a test-pattern for the cancellation theorem.

Suppose that $y \cup z = x \cup z$ and $y \cap z = x \cap z$.

It follows that $(y \cup z) \cap \tilde{z} = (x \cup z) \cap \tilde{z}$,

So, $(y \cap \tilde{z}) \cup (z \cap \tilde{z}) = (x \cap \tilde{z}) \cup (z \cap \tilde{z})$, [dpiu]

and $(y \cap \tilde{z}) \cup \emptyset = (x \cap \tilde{z}) \cup \emptyset.$ $[\forall_x\, x \cap \tilde{x} = \emptyset]$

Hence, $y \cap \tilde{z} = x \cap \tilde{z}.$ $[\forall_x\, x \cup \emptyset = x]$

But, since $y \cap z = x \cap z,$

$$(y \cap \tilde{z}) \cup (y \cap z) = (x \cap \tilde{z}) \cup (x \cap z),$$

so, $y \cup (\tilde{z} \cap z) = x \cup (\tilde{z} \cap z),$ [ℓdpui]

$$y \cup \emptyset = x \cup \emptyset, \qquad [\forall_x\, \tilde{x} \cap x = \emptyset]$$

and $y = x.$ $[\forall_x\, x \cup \emptyset = x]$

Hence, $x = y.$

Consequently, if $y \cup z = x \cup z$ and $y \cap z = x \cap z$ then $x = y$.

15. $\forall_x \forall_y \forall_z \; x \sim (y \cup z) = (x \sim y) \sim z$

[$x \cap \widetilde{y \cup z} = x \cap (\tilde{y} \cap \tilde{z}) = (x \cap \tilde{y}) \cap \tilde{z}$]

16. $\forall_x \forall_y \forall_z \; (x \sim y) \cap z = (x \cap z) \sim y$

[Use the principle of relative complements, the api and the cpi.]

17. $\forall_x \forall_y \forall_z \; (x \sim y) \cup (x \sim z) = x \sim (y \cap z)$

[$(x \cap \tilde{y}) \cup (x \cap \tilde{z}) = x \cap (\tilde{y} \cup \tilde{z}) = x \cap \widetilde{y \cap z}$]

18. $\forall_x \forall_y \forall_z \; (x \cup y) \sim z = (x \sim z) \cup (y \sim z)$

[$(x \cup y) \cap \tilde{z} = (x \cap \tilde{z}) \cup (y \cap \tilde{z})$]

19. $\forall_x \forall_y \forall_z \; (x \cap y) \sim z = (x \sim z) \cap (y \sim z)$

[$(x \cap y) \cap \tilde{z} = (x \cap \tilde{z}) \cap (y \cap \tilde{z})$]

20. $\forall_x \forall_y \forall_z \; (x \sim y) \sim z = (x \sim z) \sim (y \sim z)$

[$(x \cap \tilde{z}) \cap \widetilde{y \cap \tilde{z}} = (x \cap \tilde{z}) \cap (\tilde{y} \cup z) =$
$[(x \cap \tilde{z}) \cap \tilde{y}] \cup [(x \cap \tilde{z}) \cap z] = (x \cap \tilde{z}) \cap \tilde{y} = (x \cap \tilde{y}) \cap \tilde{z}$]

21. $\forall_x \forall_y \forall_z \; (x \sim y) \cap z = (x \cap z) \sim (y \cap z)$

[Similar to 18.]

22. $\forall_x \forall_y \forall_z$ [$z \subseteq x$ if and only if $x \sim (y \sim z) = (x \sim y) \cup z$]

[$x \sim (y \sim z) = x \cap (\tilde{y} \cup z) = (x \cap \tilde{y}) \cup (x \cap z)$. So, $x \sim (y \sim z) = (x \sim y) \cup z$ if and only if $(x \cap \tilde{y}) \cup (x \cap z) = (x \cap \tilde{y}) \cup z$. But, in any case, $(x \cap \tilde{y}) \cap (x \cap z) = (x \cap \tilde{y}) \cap z$. So, by the cancellation theorem, $x \sim (y \sim z) = (x \sim y) \cup z$ if and only if $x \cap z = z$.]

6. $\forall_x \forall_y\ y \subseteq x$ if and only if $y \cap \tilde{x} = \emptyset$

[$y = (y \cap x) \cup (y \cap \tilde{x})$, so, if $y \cap \tilde{x} = \emptyset$ then $y \cap x = y$, and $y \subseteq x$. And, if $y \subseteq x$, so that $y = y \cap x$, then $y \cap \tilde{x} = (y \cap x) \cap \tilde{x} = \ldots$.]

7. $\forall_x \forall_y\ x \cup y = x \cup (y \sim x)$

[$x \cup (y \sim x) = (x \cup y) \cap (x \cup \tilde{x})$, by the principle for relative complements and the ℓdpui.]

8. $\forall_x \forall_y\ x \sim y = (x \cup y) \sim y$

[$(x \cup y) \cap \tilde{y} = x \cap \tilde{y}$]

9. $\forall_x \forall_y\ x \sim y = x \sim (x \cap y)$

[$x \cap \widetilde{x \cap y} = x \cap (\tilde{x} \cup \tilde{y}) = x \cap \tilde{y}$]

10. $\forall_x \forall_y\ x \sim y = \tilde{x} \cup y$

[$\widetilde{x \cap \tilde{y}} = \tilde{x} \cup y$]

11. $\forall_x \forall_y\ x \sim (x \sim y) = x \cap y$

[$x \cap \widetilde{x \cap \tilde{y}} = x \cap (\tilde{x} \cup y) = x \cap y$]

12. $\forall_x \forall_y\ x \sim (y \sim x) = x$

[$x \cap (\tilde{y} \cup x) = x$ ($x \subseteq z \cup x$, for any z)]

13. $\forall_x \forall_y \forall_z\ x \sim (y \sim z) = (x \sim y) \cup (x \cap z)$

[$x \cap (\tilde{y} \cup z) = (x \cap \tilde{y}) \cup (x \cap z)$]

14. $\forall_x \forall_y$ if $x \sim y = y \sim x$ then $x = y$

[$x = (x \cap \tilde{y}) \cup (x \cap y)$, so, if $x \cap \tilde{y} = y \cap \tilde{x}$ then $x = (y \cap \tilde{x}) \cup (x \cap y) = y \cap (\tilde{x} \cup x) = y \cap S = y$]

The last two suggest that the dual of relative complementation is a sort of analogue of division. So [although this notation is not standard], one might introduce the principle:

$$\forall_x \forall_y \; x \dot{\div} y = x \cup \tilde{y}$$

This would have the advantage that we could, after introducing this principle and the principle for relative complements, obtain duals by interchanging the binary operators '$\sim$' and '$\dot{\div}$' [in addition, of course, to interchanging '$\cup$' and '$\cap$', '$\emptyset$' and 'S', and '$\subseteq$' and '$\supseteq$'].

*

Here, for the use of <u>those of your students who wish to explore further</u> into the algebra of sets, are additional theorems which can be derived from the basic principles. To save space, we give only one of each pair of dual theorems. [We also give hints for proofs.]

1. $\forall_x \forall_y \forall_z \; (x \cup y) \cup z = (x \cup z) \cup (y \cup z)$

   [Replace 'z' on the left by '$z \cup z$', and use the cpu and the apu.]

2. $\forall_x \forall_y \forall_z$ if $x \subseteq z$ and $y \subseteq z$ then $x \cup y \subseteq z$

   [If $z \cap x = x$ and $z \cap y = y$ then $(z \cap x) \cup (z \cap y) = x \cup y$. Use the ℓdpiu.]

3. $\forall_x \forall_y$ if $x \cup y = x \cap y$ then $x = y$

   [Since $x \cap y \subseteq x$, if $x \cup y = x \cap y$ then $x \cup y \subseteq x$. So, since $x \subseteq x \cup y$, $x \cup y = x$. Hence $y \subseteq x$. Etc..]

4. $\forall_x \forall_y \forall_z$ if $x \cup y = y \cap z$ then $(x \subseteq y$ and $y \subseteq z)$

   [Similar to preceding.]

5. $\forall_x \forall_y \; x = (x \cap y) \cup (x \sim y)$

   [Use the principle for relative complements, the ℓpdiu, etc..]

*

Note that, by using the inclusion principle and its dual, the theorems of Exercises 4 and 3 of Part B on page 5-21 may be transformed into:

$$\forall_x \forall_y\ (x \cup y) \cap x = x \text{ and: } \forall_x \forall_y\ (x \cap y) \cup x = x,$$

respectively. These theorems are called <u>absorption</u> <u>laws</u>.

*

Although, as noted on TC[5-18], the operation oppositing, for real numbers, has no analogue, there is, in a limited way, an analogue of subtracting. To see this, we introduce:

<u>Principle</u> <u>for</u> <u>relative</u> <u>complements</u>

$$\forall_x \forall_y\ x \sim y = x \cap \tilde{y}$$

[Read '$x \sim y$' as 'the complement of y relative to x'.]

Among the interesting theorems are:

(1) $\forall_x\ x \sim \emptyset = x$ (2) $\forall_x\ S \sim x = \tilde{x}$

(3) $\forall_x \forall_y\ x \subseteq y$ if and only if $x \sim y = \emptyset$

(4) $\forall_x \forall_y\ (x \cup y) \sim y = x$ if and only if $y \subseteq \tilde{x}$

(5) $\forall_x \forall_y\ (x \sim y) \cup y = x$ if and only if $y \subseteq x$

[Proofs are most easily obtained by first using the principle for relative complements to replace relative complements by intersections.]

The duals of these are, of course, theorems. To obtain the dual of (1), '$\forall_x\ x \sim \emptyset = x$', we first use the principle for relative complements and get '$\forall_x\ x \cap \tilde{\emptyset} = x$'. Then, we make the replacements necessary for dualizing, and get '$\forall_x\ x \cup \tilde{S} = x$'. Here are the duals for the 5 theorems:

(1′) $\forall_x\ x \cup \tilde{S} = x$ (2′) $\forall_x\ \emptyset \cup \tilde{x} = \tilde{x}$

(3′) $\forall_x \forall_y\ y \subseteq x$ if and only if $x \cup \tilde{y} = S$

(4′) $\forall_x \forall_y\ (x \cap y) \cup \tilde{y} = x$ if and only if $\tilde{x} \subseteq y$

(5′) $\forall_x \forall_y\ (x \cup \tilde{y}) \cap y = x$ if and only if $x \subseteq y$

Since $y \cap x = x$,

it follows that $z \cap x = x$.

But, if $z \cap x = x$ then $x \subseteq z$. $[\forall_x \forall_y$ if $x \cap y = y$ then $y \subseteq x]$

Hence, $x \subseteq z$.

Consequently, if $x \subseteq y$ and $y \subseteq z$ then $x \subseteq z$.

*

As noted on TC[5-17, 18], in dualizing a sentence which contains '$\subseteq$', one replaces this symbol by '$\supseteq$' [or, replaces, for example, '$y \subseteq x$' by '$x \subseteq y$']. With this extension of the notion of dual, to show that the dual of a theorem is a theorem it is sufficient to show that the dual of the inclusion principle is a theorem. That is, it is sufficient to show that the following generalization is a theorem:

$$\forall_x \forall_y \; x \subseteq y \text{ if and only if } x \cup y = y$$

Here is a test-pattern.

Suppose that $x \cup y = y$.

Since $x \subseteq x \cup y$, $[\forall_x \forall_y \; x \subseteq x \cup y]$

it follows that $x \subseteq y$.

Consequently, if $x \cup y = y$ then $x \subseteq y$.

Suppose that $x \subseteq y$.

If $x \subseteq y$ then $y \cap x = x$. $[\forall_x \forall_y$ if $y \subseteq x$ then $x \cap y = y]$

Consequently, $y \cap x = x$.

Hence, $x \cup y = (y \cap x) \cup y$

$= (y \cap x) \cup (y \cap S)$ $[\forall_x \; x \cap S = x]$

$= y \cap (x \cup S)$ $[\forall_x \forall_y \forall_z \; x \cap (y \cup z) = (x \cap y) \cup (x \cap z)]$

$= y \cap S$ $[\forall_x \; x \cup S = S]$

$= y.$ $[\forall_x \; x \cap S = x]$

Consequently, if $x \subseteq y$ then $x \cup y = y$.

Hence, if $x \cup y = y$ then $x \subseteq y$ and if $x \subseteq y$ then $x \cup y = y$ --that is, $x \subseteq y$ if and only if $x \cup y = y$.

> <u>Correction.</u> On page 5-22, the last part of Theorem 5 should be:
>
> then $\tilde{x} = y$

Answers for Part B.

1. $x \cap \emptyset = \emptyset$ $\quad [\forall_x\ x \cap \emptyset = \emptyset]$

   if $x \cap \emptyset = \emptyset$ then $\emptyset \subseteq x$ $\quad [\forall_x \forall_y$ if $x \cap y = y$ then $y \subseteq x]$

   $\emptyset \subseteq x$

2. $S \cap x = x$ $\quad [\forall_x\ S \cap x = x]$

   if $S \cap x = x$ then $x \subseteq S$ $\quad [\forall_x \forall_y$ if $x \cap y = y$ then $y \subseteq x]$

   $x \subseteq S$

3. $x \cap (x \cap y) = (x \cap x) \cap y$ $\quad [\forall_x \forall_y \forall_z\ (x \cap y) \cap z = x \cap (y \cap z)]$

   $= x \cap y$ $\quad [\forall_x\ x \cap x = x]$

   if $x \cap (x \cap y) = x \cap y$ then $x \cap y \subseteq x$ $\quad [\forall_x \forall_y$ if $x \cap y = y$ then $y \subseteq x]$

   $x \cap y \subseteq x$

4. $(x \cup y) \cap x = (x \cup y) \cap (x \cup \emptyset)$ $\quad [\forall_x\ x \cup \emptyset = x]$

   $= x \cup (y \cap \emptyset)$ $\quad [\forall_x \forall_y \forall_z\ x \cup (y \cap z) = (x \cup y) \cap (x \cup z)]$

   $= x \cup \emptyset$ $\quad [\forall_x\ x \cap \emptyset = \emptyset]$

   $= x$ $\quad [\forall_x\ x \cup \emptyset = x]$

   if $(x \cup y) \cap x = x$ then $x \subseteq x \cup y$ $\quad [\forall_x \forall_y$ if $x \cap y = y$ then $y \subseteq x]$

   $x \subseteq x \cup y$

5. Suppose that $x \subseteq y$ and $y \subseteq z$.

   Then, $\quad y \cap x = x$ and $z \cap y = y$.

   Since, $\quad z \cap y = y$,

   $(z \cap y) \cap x = y \cap x$,

   and $\quad z \cap (y \cap x) = y \cap x$. $\quad [\forall_x \forall_y \forall_z\ (x \cap y) \cap z = x \cap (y \cap z)]$

*

Open-Book Quiz.

Use the basic principles and the theorems on pages 5-22 and 5-23 to do the following exercises.

1. Simplify.

   (a) $\widetilde{A} \cap \widetilde{(B \cup C)}$ (b) $(A \cup \widetilde{A}) \cap (A \cap \widetilde{A})$

2. (a) Of what theorem [Theorem __?__] is the sentence:

   $$A \subseteq A \cup B$$

   an instance?

   (b) Is it also the case that $B \subseteq A \cup B$?

   (c) From what theorem [Theorem __?__] does the sentence:

   $$\text{if } C \subseteq A \text{ then } C \cup A = A$$

   follow immediately?

   (d) Simplify: $(A \cap B) \cup A$

*

Answers for Quiz.

1. (a) $A \cup (B \cup C)$ (b) $\emptyset$

2. (a) Theorem 10(a) (b) Yes [Commutativity and Theorem 10(a)]

   (c) Theorem 11 [only-if part]

   (d) A[Theorems 10(b) and 3(a)]

In order to derive the inclusion principle from P1 through P7 it is convenient to use a stronger result than P7*:

$$\text{P7**} \qquad \forall_e [e \in x \Longleftrightarrow e \in y] \Longleftrightarrow x = y$$

Since we have already derived P7*, all that remains to do to derive P7** is to derive:

$$\text{if } x = y \text{ then } \forall_e [e \in x \Longleftrightarrow e \in y]$$

To do so, suppose that $x = y$.

Since $\quad e \in x \Longleftrightarrow e \in x,$

it follows that $\quad e \in x \Longleftrightarrow e \in y.$

So, $\quad \forall_e [e \in x \Longleftrightarrow e \in y].$

Hence, $\quad$ if $x = y$ then $\forall_e [e \in x \Longleftrightarrow e \in y].$

In deriving the inclusion principle we shall make use of a principle of logic which assures us that the sentence:

$$\forall_e (\text{if } e \in y \text{ then } e \in x) \Longleftrightarrow \forall_e [(e \in x \text{ and } e \in y) \Longleftrightarrow e \in y]$$

is valid. [Although the validity of this sentence can be justified on the basis of simple principles of logic with which we are already acquainted, to do so would take us too far afield.]

Here is a derivation of the inclusion principle:

$$\left.\begin{array}{l} y \subseteq x \\ \Longleftrightarrow \forall_e \text{ if } e \in y \text{ then } e \in x \end{array}\right\} [\text{P6}]$$

$$\Longleftrightarrow \forall_e [(e \in x \text{ and } e \in y) \Longleftrightarrow e \in y]$$

So, $\quad y \subseteq x \Longleftrightarrow \forall_e [(e \in x \text{ and } e \in y) \Longleftrightarrow e \in y].$

But, $\quad e \in x \cap y \Longleftrightarrow (e \in x \text{ and } e \in y). \quad$ [P2]

So, $\quad y \subseteq x \Longleftrightarrow \forall_e [e \in x \cap y \Longleftrightarrow e \in y].$

But, $\forall_e [e \in x \cap y \Longleftrightarrow e \in y] \Longleftrightarrow x \cap y = y. \quad$ [P7**]

So, $\quad y \subseteq x \Longleftrightarrow x \cap y = y.$

Therefore, $\quad \forall_x \forall_y [y \subseteq x \Longleftrightarrow x \cap y = y].$

Here is a derivation of the intersecting principle for complements:

$$e \notin \emptyset \qquad [P4]$$

Hence, if $e \notin \tilde{x} \cap x$ then $e \notin \emptyset$.

Consequently, if $e \in \emptyset$ then $e \in \tilde{x} \cap x$.

$$\begin{aligned} & e \in \tilde{x} \cap x \\ & \Longleftrightarrow (e \in \tilde{x} \text{ and } e \in x) && [P2] \\ & \Longleftrightarrow (e \notin x \text{ and } e \in x) && [P3] \end{aligned}$$

So, $e \in \tilde{x} \cap x \Longleftrightarrow (e \notin x \text{ and } e \in x)$.

But, it is not the case that $(e \notin x \text{ and } e \in x)$.

So, $e \notin \tilde{x} \cap x$.

Hence, if $e \notin \emptyset$ then $e \notin \tilde{x} \cap x$.

Consequently, if $e \in \tilde{x} \cap x$ then $e \in \emptyset$.

So, $e \in \tilde{x} \cap x \Longleftrightarrow e \in \emptyset$.

Therefore, $\forall_e\, [e \in \tilde{x} \cap x \Longleftrightarrow e \in \emptyset]$.

But, if $\forall_e\, [e \in \tilde{x} \cap x \Longleftrightarrow e \in \emptyset]$ then $\tilde{x} \cap x = \emptyset$. [P7*]

So, $\tilde{x} \cap x = \emptyset$.

Therefore, $\forall_x\, \tilde{x} \cap x = \emptyset$.

[Each of the two conditionals, above, which are preceded by 'Consequently,' is a consequence of the conditional directly above it by virtue of the principle of logic by which a conditional sentence is a consequence of its contrapositive.]

Here is a derivation of the unioning principle for complements:

$$
\begin{array}{lrll}
 & & e \in \tilde{x} \cup x & \\
 & \Longleftrightarrow & (e \in \tilde{x} \text{ or } e \in x) & \text{P1} \\
 & \Longleftrightarrow & (e \notin x \text{ or } e \in x) & \text{P3} \\
\text{So,} & e \in \tilde{x} \cup x \Longleftrightarrow & (e \notin x \text{ or } e \in x). & \\
\text{But,} & & e \notin x \text{ or } e \in x. & \\
\text{So,} & & e \in \tilde{x} \cup x. & \\
\text{Hence,} & \text{if } e \in S \text{ then} & e \in \tilde{x} \cup x. & \\
\text{Also,} & & e \in S. & [\text{P5}] \\
\text{Hence,} & \text{if } e \in \tilde{x} \cup x & \text{then } e \in S. & \\
\text{Consequently,} & e \in \tilde{x} \cup x \Longleftrightarrow & e \in S. & \\
\text{Therefore,} & \forall_e [e \in \tilde{x} \cup x \Longleftrightarrow & e \in S]. & \\
\text{But,} & \text{if } \forall_e [e \in \tilde{x} \cup x \Longleftrightarrow & e \in S] \text{ then } \tilde{x} \cup x = S. & [\text{P7*}] \\
\text{So,} & \tilde{x} \cup x = S. & & \\
\text{Therefore,} & \forall_x \ \tilde{x} \cup x = S. & &
\end{array}
$$

[Each of the two conditionals above, which are preceded by 'Hence', is a consequence of the sentence directly above it. The principle of logic appealed to here is the principle according to which a conditional sentence is implied by its consequent.]

Therefore, $\forall_e\ [e \in x \cup \emptyset \iff e \in x]$.

But, if $\forall_e\ [e \in x \cup \emptyset \iff e \in x]$ then $x \cup \emptyset = x$. [P7*]

So, $x \cup \emptyset = x$.

Hence, $\forall_x\ x \cup \emptyset = x$.

Here is a derivation of the principle for intersecting with S:

Suppose that $e \in x$.

Since $e \in S$,

$e \in x$ and $e \in S$.

Since $e \in x \cap S \iff (e \in x$ and $e \in S)$, [P2]

$e \in x \cap S$.

Hence, if $e \in x$ then $e \in x \cap S$.

Suppose that $e \in x \cap S$.

Since $e \in x \cap S \iff (e \in x$ and $e \in S)$, [P2]

$e \in x$ and $e \in S$.

So, $e \in x$.

Hence, if $e \in x \cap S$ then $e \in x$.

Consequently, $e \in x \cap S \iff e \in x$.

Therefore, $\forall_e\ [e \in x \cap S \iff e \in x]$

But, if $\forall_e\ [e \in x \cap S \iff e \in x]$ then $x \cap S = x$. [P7*]

So, $x \cap S = x$.

Hence, $\forall_x\ x \cap S = x$.

$$\Longleftrightarrow ((e \in x \text{ and } e \in z) \text{ or } (e \in y \text{ and } e \in z)) \quad \text{P2}$$

$$\Longleftrightarrow (e \in x \cap z \text{ or } e \in y \cap z) \quad \text{P1}$$

$$\Longleftrightarrow e \in (x \cap z) \cup (y \cap z)$$

So, $e \in (x \cup y) \cap z \Longleftrightarrow e \in (x \cap z) \cup (y \cap z).$

Hence, $\forall_e[e \in (x \cup y) \cap z \Longleftrightarrow e \in (x \cap z) \cup (y \cap z)].$

But, if $\forall_e[e \in (x \cup y) \cap z \Longleftrightarrow e \in (x \cap z) \cup (y \cap z)]$ then

$$(x \cup y) \cap z = (x \cap z) \cup (y \cap z). \qquad [P7*]$$

So, $(x \cup y) \cap z = (x \cap z) \cup (y \cap z).$

Hence, $\forall_x \forall_y \forall_z (x \cup y) \cap z = (x \cap z) \cup (y \cap z).$

Here is a derivation of the principle for unioning with $\emptyset$:

Suppose that $e \in x.$

Then $e \in x \text{ or } e \in \emptyset.$

Since $e \in x \cup \emptyset \Longleftrightarrow (e \in x \text{ or } e \in \emptyset),$ [P1]

$e \in x \cup \emptyset.$

Hence, if $e \in x$ then $e \in x \cup \emptyset.$

Suppose that $e \in x \cup \emptyset.$

Since $e \in x \cup \emptyset \Longleftrightarrow (e \in x \text{ or } e \in \emptyset),$ [P1]

$e \in x \text{ or } e \in \emptyset.$

But $e \notin \emptyset.$

So, $e \in x.$

Hence, if $e \in x \cup \emptyset$ then $e \in x.$

Consequently, $e \in x \cup \emptyset \Longleftrightarrow e \in x.$

A derivation of the commutative principle for intersecting can be obtained from the derivation above merely by replacing '$\cup$' by '$\cap$', 'or' by 'and', and 'P1' by 'P2'. Line (2) in the derivation so obtained is the logically valid sentence '$(e \in x \text{ and } e \in y) \Longleftrightarrow (e \in y \text{ and } e \in x)$'.

The associative principle for unioning is derived as follows:

$$\begin{array}{lll}
 & e \in (x \cup y) \cup z & \\
\Longleftrightarrow & (e \in x \cup y \text{ or } e \in z) & \} \text{ P1} \\
\Longleftrightarrow & ((e \in x \text{ or } e \in y) \text{ or } e \in z) & \} \text{ P1} \\
\Longleftrightarrow & (e \in x \text{ or } (e \in y \text{ or } e \in z)) & \} \text{ [Why?]} \\
\Longleftrightarrow & (e \in x \text{ or } e \in y \cup x) & \} \text{ P1} \\
\Longleftrightarrow & e \in x \cup (y \cup z) & \} \text{ P1}
\end{array}$$

So, $e \in (x \cup y) \cup z \Longleftrightarrow e \in x \cup (y \cup z)$.

Hence, $\forall_e\, [e \in (x \cup y) \cup z \Longleftrightarrow e \in x \cup (y \cup z)]$.

But, if $\forall_e\, [e \in (x \cup y) \cup z \Longleftrightarrow e \in x \cup (y \cup z)]$ then

$$(x \cup y) \cup z = x \cup (y \cup z). \qquad \text{[P7*]}$$

So, $(x \cup y) \cup z = x \cup (y \cup z)$.

Hence, $\forall_x \forall_y (x \cup y) \cup z = x \cup (y \cup z)$.

A derivation of the associative principle for intersecting can be obtained from the above derivation by replacing '$\cup$' by '$\cap$', 'or' by 'and', and 'P1' by 'P2'.

Here is a derivation of the distributive principle for intersecting over unioning:

$$\begin{array}{lll}
 & e \in (x \cup y) \cap z & \\
\Longleftrightarrow & (e \in (x \cup y) \text{ and } e \in z) & \} \text{ P2} \\
\Longleftrightarrow & ((e \in x \text{ or } e \in y) \text{ and } e \in z) & \} \text{ P1}
\end{array}$$

Here is a derivation of the commutative principle for unioning:

$$\left.\begin{array}{ll} & e \in x \cup y \\ (1) & \Longleftrightarrow (e \in x \text{ or } e \in y) \end{array}\right\} \text{P1}$$

$$\left.\begin{array}{ll} (2) & \Longleftrightarrow (e \in y \text{ or } e \in x) \\ (3) & \Longleftrightarrow e \in y \cup x \end{array}\right\} \text{P1}$$

(4) So, $\quad e \in x \cup y \Longleftrightarrow e \in y \cup x.$

(5) Hence, $\quad \forall_e\, [e \in x \cup y \Longleftrightarrow e \in y \cup x].$

(6) But, $\quad$ if $\forall_e\, [e \in x \cup y \Longleftrightarrow e \in y \cup x]$ then $x \cup y = y \cup x.$ [P7*]

(7) Hence, $\quad x \cup y = y \cup x.$

(8) Consequently, $\forall_x \forall_y\, x \cup y = y \cup x.$

Line (1) is a consequence of P1. Line (2) is the logically valid sentence '$(e \in x \text{ or } e \in y) \Longleftrightarrow (e \in y \text{ or } e \in x)$'. Line (3) is a consequence of P1, and line (4) follows from the first three lines by virtue of the substitution principle for biconditionals. Schematically:

$$\dfrac{\text{line (3)}\;\; \overset{\text{P1}}{\overline{r \Longleftrightarrow s}} \qquad \dfrac{\text{line (2)}\;\; q \Longleftrightarrow r \qquad \overset{\text{P1}}{\overline{p \Longleftrightarrow q}}\;\; \text{line (1)}}{p \Longleftrightarrow r}}{p \Longleftrightarrow s \;\; \text{line (4)}}$$

Line (5) is justified by the fact that the preceding lines constitute a pattern which shows that '$\forall_e\, [e \in x \cup y \Longleftrightarrow e \in y \cup x]$' is a consequence of P1. Line (6) is a consequence of P7*. Line (7) follows lines (5) and (6) by modus ponens, another principle of logic, this time one dealing with conditional sentences. Finally, line (8) is justified by the fact that lines (5), (6), and (7) constitute a pattern which shows that the generalization '$\forall_x \forall_y\, x \cup y = y \cup x$' is a consequence of line (5) and P7*.

principle for biconditional sentences. Using the tree-form of presenting derivations we can schematize these steps in the derivation as follows:

$$\dfrac{\text{line (2)}\ \dfrac{P6}{q \Longleftrightarrow s} \qquad \dfrac{\dfrac{P6}{p \Longleftrightarrow r}\ \text{line (1)} \qquad (p \text{ and } q) \Longleftrightarrow (p \text{ and } q)\ \text{line (3)}}{(p \text{ and } q) \Longleftrightarrow (r \text{ and } q)}}{(p \text{ and } q) \Longleftrightarrow (r \text{ and } s)\ \text{line (4)}}$$

In deriving line (5) from line (4) we use a logical principle concerning conjunctions and universal generalizations according to which a biconditional sentence of the form '$[\forall_e\ p(e)]$ and $[\forall_e\ q(e)] \Longleftrightarrow \forall_e[p(e)$ and $q(e)]$' is valid merely on logical grounds. Line (6) follows from line (5) because '$e \in x \Longleftrightarrow e \in y$' is just an abbreviation for '(if $e \in y$ then $e \in x$) and (if $e \in x$ then $e \in y$)'. Line (7) sums up lines (4), (5), and (6), according to the scheme below.

$$\dfrac{\text{line (6)}\ c \Longleftrightarrow d \qquad \dfrac{\text{line (5)}\ b \Longleftrightarrow c \qquad a \Longleftrightarrow b\ \text{line (4)}}{a \Longleftrightarrow c}}{a \Longleftrightarrow d\ \text{line (7)}}$$

Line (8) is a consequence of P7, and line (9) is obtained from this and line (7) by the substitution principle for biconditionals. Line (10) is justified by the fact that the preceding lines constitute a pattern [compare page 2-33 of Unit 2] which shows that the generalization '$\forall_x \forall_y$ if $\forall_e [e \in x \Longleftrightarrow e \in y]$ then $x = y$' is a consequence of P6 and P7. [When we use this generalization in later proofs we shall refer to it by 'P7*'.]

*

We can now exhibit derivations of the basic principles, on page 5-22, from P1 through P7. [Bearing in mind the analogy between '$\Longleftrightarrow$' and '=', it is clear that we can adopt a form analogous to that introduced on page 2-35 of Unit 2.]

So, one can use a biconditional sentence to justify replacing one <u>sentence</u> by another just as he uses an equation as a premiss to justify his replacing one <u>expression</u> by another.

As an illustration of how these logical principles are used, we shall derive:

$$\forall_x \forall_y \text{ if } \forall_e \text{ [e } \epsilon \text{ x if and only if e } \epsilon \text{ y] then x = y}$$

from P6 and P7. [To save space, we shall abbreviate 'if and only if' by '$\Longleftrightarrow$'.]

(1) $y \subseteq x \Longleftrightarrow \forall_e$ if $e \epsilon y$ then $e \epsilon x$ [P6]

(2) $x \subseteq y \Longleftrightarrow \forall_e$ if $e \epsilon x$ then $e \epsilon y$ [P6]

(3) So, since $y \subseteq x$ and $x \subseteq y \Longleftrightarrow y \subseteq x$ and $x \subseteq y$,

(4) $y \subseteq x$ and $x \subseteq y \Longleftrightarrow [\forall_e$ if $e \epsilon y$ then $e \epsilon x]$ and $[\forall_e$ if $e \epsilon x$ then $e \epsilon y]$

(5) $\Longleftrightarrow \forall_e$ [(if $e \epsilon y$ then $e \epsilon x$) and (if $e \epsilon x$ then $e \epsilon y$)]

(6) $\Longleftrightarrow \forall_e [e \epsilon x \Longleftrightarrow e \epsilon y]$.

(7) So, $y \subseteq x$ and $x \subseteq y \Longleftrightarrow \forall_e [e \epsilon x \Longleftrightarrow e \epsilon y]$.

(8) But, if $y \subseteq x$ and $x \subseteq y$ then $x = y$. [P7]

(9) So, if $\forall_e [e \epsilon x \Longleftrightarrow e \epsilon y]$ then $x = y$.

(10) Consequently, $\forall_x \forall_y$ if $\forall_e [e \epsilon x \Longleftrightarrow e \epsilon y]$ then $x = y$.

In the derivation above, lines (1) and (2) are consequences of P6, and line (3), since it is of the form '$p \Longleftrightarrow p$', is valid on merely logical grounds. Line (4) then follows from the first three lines by virtue of the substitution

So, one principle of logic concerning biconditionals says that from a biconditional sentence one can infer either of two conditional sentences:

$$\frac{\text{p if and only if q}}{\text{if q then p}}, \quad \text{or:} \quad \frac{\text{p if and only if q}}{\text{if p then q}}$$

[This principle of logic has already been used on page 5-21.]

And, a second principle of logic says that if a conditional sentence and its converse have both been derived from given premisses, then the related biconditional sentence is also a consequence of these premisses. For short:

$$\frac{\text{if q then p} \qquad \text{if p then q}}{\text{p if and only if q}}$$

The principal value of biconditional sentences stems from an analogy between such sentences and equations. As explained on TC[1-56]a and b, and illustrated on TC[2-31, 32]b, TC[2-64]a and b, and TC[2-65]b and c, the logical principles which govern the use of '=' all follow from the fact that '$\forall_x\ x = x$' is valid merely on logical grounds, and from the following substitution principle for '=':

> From an equation and a second sentence one can infer any sentence which can be obtained by replacing an occurrence in the second sentence of either side of the equation by the other side of the equation. [Example: From the equation '$3 = 2 + 1$' and the sentence '$a < 3$', one can infer the sentence '$a < 2 + 1$'.]

Now, any sentence of the form 'p if and only if p' is valid merely on logical grounds, and there is a substitution principle for 'if and only if':

> From a biconditional sentence and a second sentence one can infer any sentence which can be obtained by replacing an occurrence in the second sentence of either component of the biconditional sentence by the other component of the biconditional sentence. [Example: From the biconditional sentence '$e \in y \cup z$ if and only if ($e \in y$ or $e \in z$)' and the sentence '$e \in x$ and $e \in y \cup z$', one can infer the sentence '$e \in x$ and ($e \in y$ or $e \in z$)'.]

For completeness, and to illustrate some techniques of proof, we show here how to derive the basic principles on page 5-22 from the alternative principles, P1 through P7, given on page 5-23. Since the new basic principles relate unions, intersections, and complements of sets to alternations ['or'], conjunctions ['and'], and denials ['not'] of sentences, it turns out that each of the basic principles of 5-22 is derived from the new ones by using an appropriate principle of logic which concerns sentences and looks much like the basic principle which is being derived. For example, to derive:

$$\forall_x \forall_y \; x \cup y = y \cup x,$$

one uses the principle of logic according to which each sentence of the form:

[p or q] if and only if [q or p]

is valid merely on logical grounds, and, so, is acceptable without need of proof. To a beginner, whose understanding of logic is, as yet, shallow, such logical principles are likely to appear trivial, and their use in deriving the corresponding principles about sets may seem to beg the question. Unfortunately, the charge of triviality can be countered only by a rather deep and lengthy analysis [which would be out of place here] of the meanings of such words as 'or', 'and', 'not', and 'if - then'. Once this has been accomplished, it is readily seen that the derivations given are not circular; naturally, if one uses the logical word 'or' in defining '$\cup$', he must expect to use principles of logic concerning alternation sentences in proving theorems about the operation of unioning.

*

In order to make use of P1 through P7 in deriving the basic principles on page 5-22, we shall need to use some logical principles concerning biconditional sentences--sentences of the form 'p if and only if q'. Such sentences can be considered to be abbreviations of sentences of the form:

(if q then p) and (if p then q)

A biconditional sentence is just an abbreviation for the conjunction of a conditional sentence and its converse.

Section 5.03 provides the student with a few opportunities to apply what he has learned about relations. Since the subject matter is geometric, we cannot presuppose more geometric knowledge than what was learned or used in Units 2 and 3. But, we do expect that students will pick up additional geometric information in this section. Such information will be called upon in exercises throughout the remainder of Unit 5.

*

As mentioned in earlier units, measures of geometric entities are numbers [in particular, numbers of arithmetic]. A side of a triangle is a segment, and one of the properties of a segment is its length. The length of a given segment might be called '3 feet', or '36 inches', or '1 yard', or ... depending upon the unit. We say that the foot-measure of the length of this segment is 3, that its inch-measure is 36, that its yard-measure is 1, ... Note the difference in terminology:

the length of the segment is 3 feet

and:

the foot-measure [of the length] of the segment is 3

Clearly, it makes no sense to talk about <u>the</u> measure of a given segment because a segment has many measures each depending upon the unit selected as the unit of measure. However, when the unit [say, the inch] has been specified somewhere in context, it is customary to use 'the measure' as an elision [say, for 'the inch-measure']. [See the third sentence of the first paragraph on page 5-26 for an application of this convention.]

*

If students do not already know how to use a compass to construct a triangle whose side-lengths are given, this is the appropriate time to teach them to do so.

*

Note that on page 5-25 we call attention to the isomorphism between the system of numbers of arithmetic and the system of nonnegative real numbers. It will be very convenient to use the nonnegative real numbers as measures of geometric entities, and to depend upon the isomorphism to translate the results thus obtained in terms of nonzero numbers of arithmetic.

*

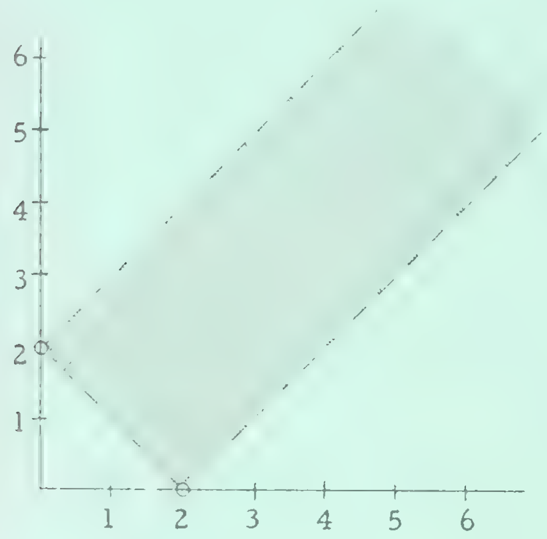
When giving the side-measures [top of page 5-25] of a triangle, we use a counterclockwise orientation, as mentioned. A clockwise orientation would serve just as well except that the counterclockwise one is customary. [Just as going to the right is the customary orientation for the positive direction.] It is essential to pick one orientation, and to stick with it so that each triangle can be associated with only one ordered pair. This need is made manifest in the discussion on page 5-26. Without a preferred orientation, the triangle pictured could be said to correspond with both (3, 4.7) and (4.7, 3). With the counterclockwise orientation it corresponds just with (3, 4.7).

Instead of drawing a second triangle corresponding with (4.7, 3), a student might say: Just turn the drawing over. Certainly, if the given triangle was pictured on a page whose reverse side was blank, one could see a picture of the second triangle by following this suggestion. Also, one could see it by carrying out our suggestion in the bracketed remark about reflection.

After establishing that the relation in question is symmetric, we next try to find out if the relation is reflexive. This one is not because it does not contain pairs such as (1, 1), (0.5, 0.5), and (0.28, 0.28). The properties of symmetry and reflexivity are discussed in greater detail in section 5.04. The reference to them on page 5-27 is just an appetite whetter.

*

The completed graph should look like this:



Compare this with the answer for the exercise in Part B on page 5-12.

*

Answers for Part A [on pages 5-28 and 5-29].

1. [The graph is like the one given above, but with the scale changed so that the corners are at (0, 5) and (5, 0).]

2. $\{(x, y): |x - 5| < y < x + 5\}$

Answers for Part D [on page 5-29].

The maximum distance obtainable between P and Q is 10, and this occurs when just P, R, and Q are collinear. [In that case, the figure is a triangle, not a quadrilateral.] The minimum distance obtainable between P and Q is 2. [In that case, Q, P, R, and S are collinear.] Thus, the measure of the segment $\overline{PQ}$ can be d if and only if $2 \leq d \leq 10$. [Challenge students to answer these questions when the measure of $\overline{PR}$ is, say, 3.]

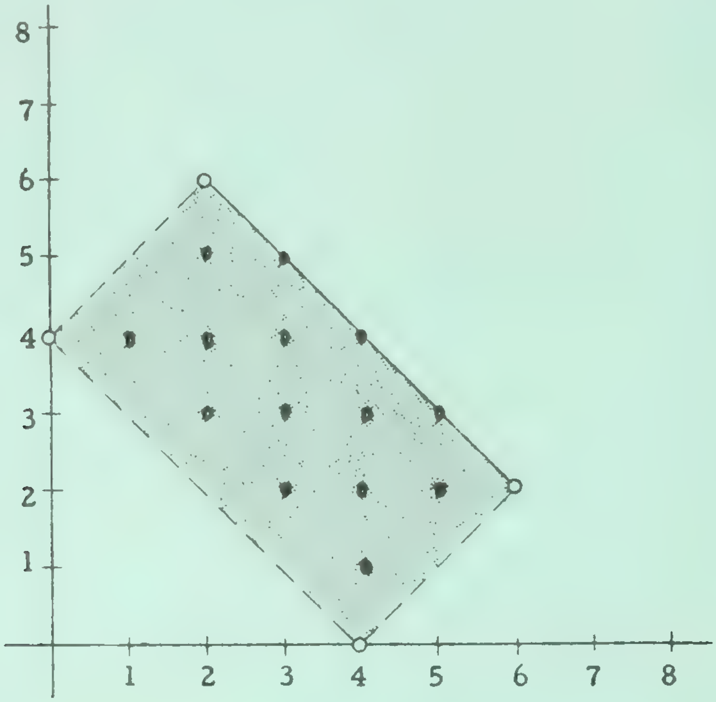
*

Answers for Part E [on page 5-29].

1. No; $2 + 5 \ngtr 8$	2. No; $9 + 11 \ngtr 20$	3. No; $8 + 8 \ngtr 16$
4. Yes	5. Yes	6. Yes
7. No; $1 + 2 \ngtr 3$	8. Yes	9. No; $6 + 4 \ngtr 12$

Answers for Part ☆C [on page 5-29].

1.



2. The graphs of points corresponding with the triangles which have inch-perimeter not exceeding 12, one side of inch-measure 4, and the other sides having whole numbers as inch-measures are shown on the figure for Exercise 1. However, not all such triangles are differently-shaped. Those which have the same set of side-measures are considered to have the same shape. Hence, there are just eight triangles which meet the conditions of Exercise 2. They are the triangles whose side-measures are the components of the ordered triples (3, 2, 4), (3, 3, 4), (4, 1, 4), (4, 2, 4), (4, 3, 4), (4, 4, 4), (5, 2, 4), and (5, 3, 4). Notice that symmetry helps in eliminating duplications.

A brace-notation name for the relation referred to in Exercise 1 is:

$$\{(x, y): \ |x - y| < 4 < x + y \leq 12 - 4\}$$

*

3. (b) $2 < n < 12$

(c) $14 < q < 18$ [or: $|16 - q| < 2 < 16 + q$]

(d) $40 < r < 80$ [or: $|20 - r| < 60 < 20 + r$]

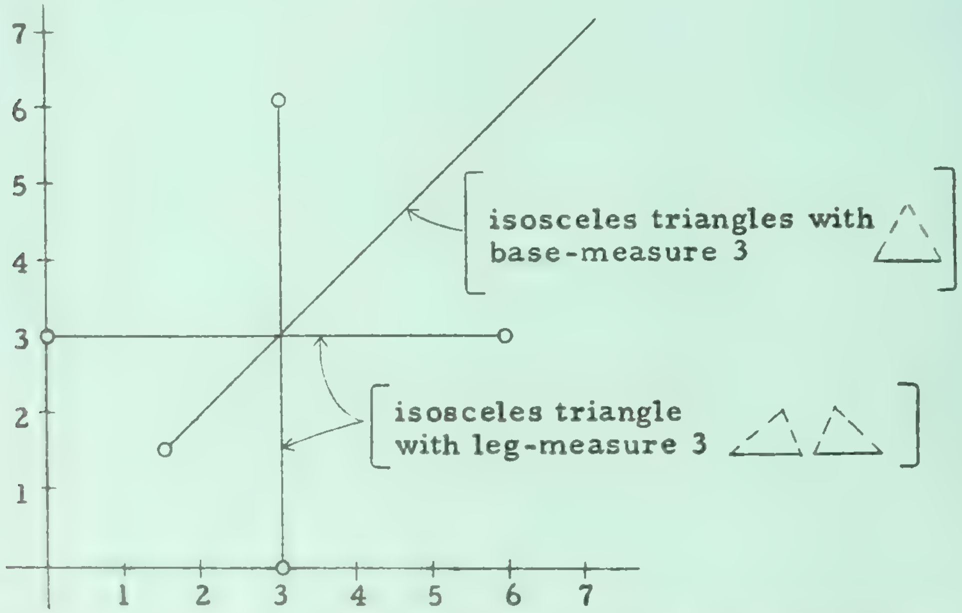
(e) $|a - b| < 11 < a + b$ [or: $|11 - b| < a < 11 + b$]

(f) $|x - z| < y < x + z$

[There are, of course, at least three correct answers for each part of Exercise 3.]

*

Answer for Part ☆B.

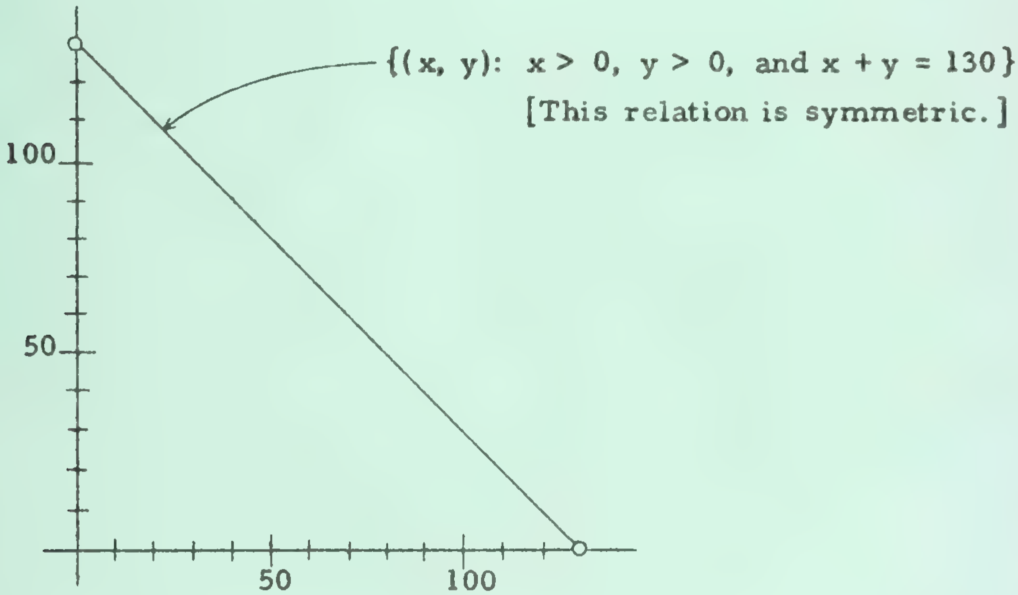


Note that this relation is a subset of the relation of the measure of one side to the measure of a second side of a triangle whose third side measures 3. Clearly, the set of all isosceles triangles each of which has a side of measure 3 is a subset of the set of all triangles each of which has a side of measure 3. Note also that the intersection of the two intervals and the half-line contains the point which corresponds with the equilateral triangle of side-measure 3.

*

Answers for Part F.

[Students are supposed to discover that the sum of the degree-measures of the other two angles is 130. The fact that students will get the same relation regardless of the side-lengths shows that the sum of the angle-measures is independent of the lengths of the sides.]



Answers for Part G [which begins on page 5-30].

1. $\{(x, y): x > 0, y > 0, \text{ and } x + y = 120\}$

2. (a) 100° (b) 20° (c) 90° (d) 10°
   (e) 60° (f) 81° (g) 80° (h) not a triangle

3. Students should sketch the interval whose end points are (0, 100) and (100, 0). A brace-notation name for this relation is: $\{(x, y): x > 0, y > 0, \text{ and } x + y = 100\}$

4. (a) (50, 50), (80, 20), (20, 80); (60, 60) [Students are supposed to be aware of the fact that two angles of a triangle have the same measure if and only if the triangle is isosceles. [Avoid saying 'equal angles'.] The fact that there is only one point of the 60°-relation which corresponds to an isosceles triangle shows that if one angle of an isosceles triangle is an angle of 60° then so are all of them.]

   (b) (60, 60) [This illustrates the fact that a triangle is equilateral if and only if each of its angles is an angle of 60°.]

5. (a) $(33\frac{1}{3}, 66\frac{2}{3})$, $(66\frac{2}{3}, 33\frac{1}{3})$, (40, 60), (60, 40);

   $(33\frac{1}{3}, 66\frac{2}{3}, 80)$, (40, 60, 80)

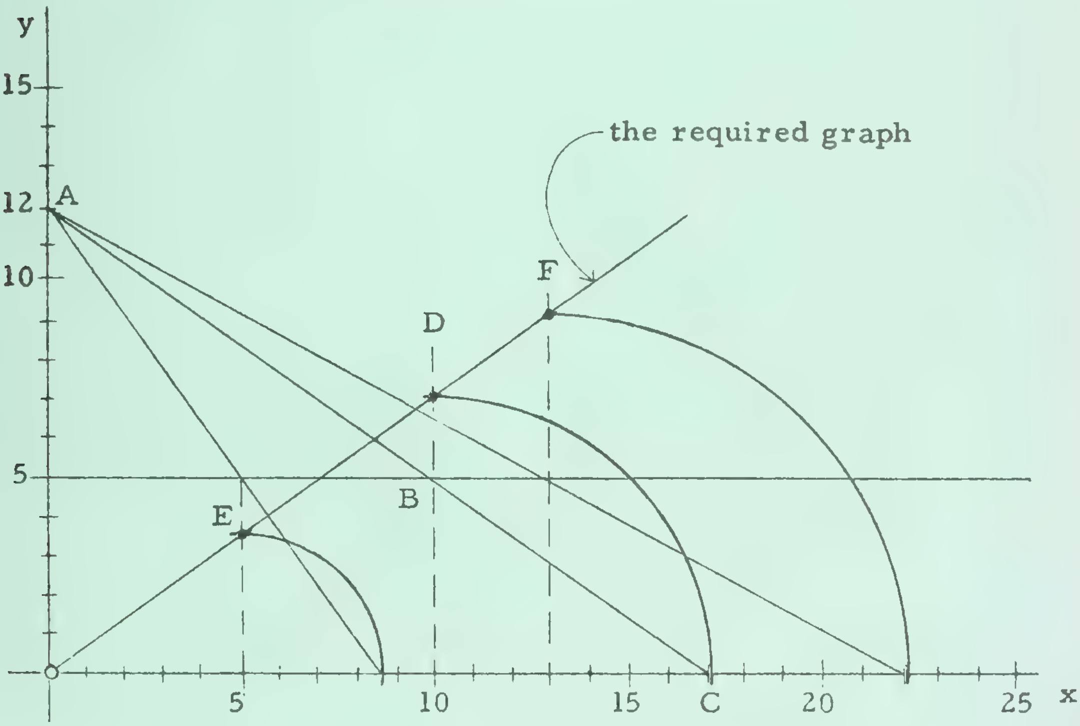
   (b) (40, 80), (80, 40), (30, 90), (90, 30);

   (40, 60, 80), (30, 60, 90)

6. (a) 30° (b) 60° (c) 28° (d) 29°

Answers for Part ☆H.

1. Students may find it helpful to develop a graphical technique for finding points which belong to this relation, a technique which does not require them to make measurements in order to find the components of the points. One such technique is shown in the diagram. [Of course, students should use cross-section paper.]



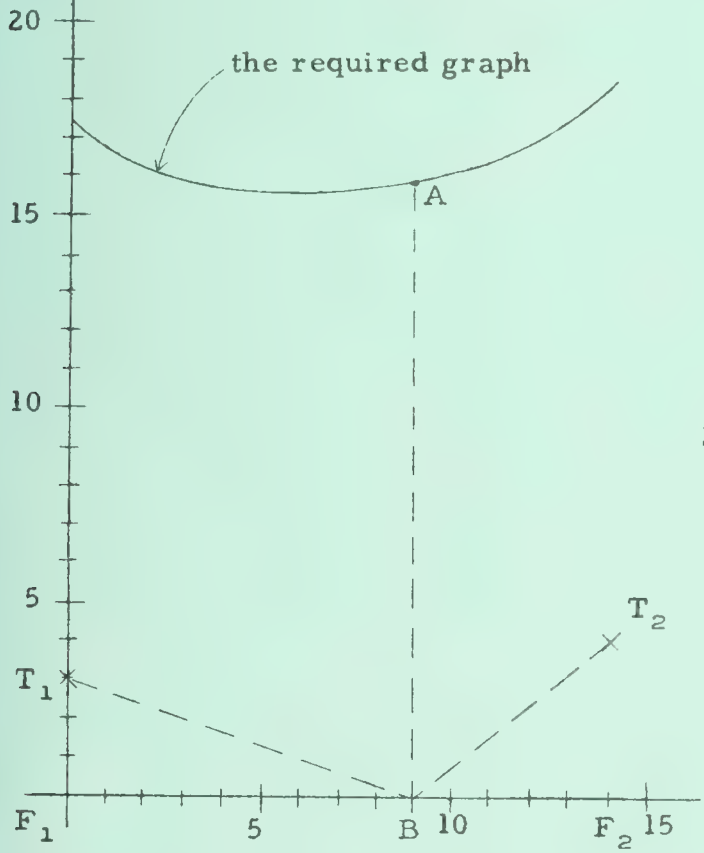
Here is how point D of the graph was located. First, the ray $\overrightarrow{AB}$ was drawn intersecting the x-axis in the point C. [Note that A is 12 units above the horizontal and B is 5 units above the horizontal.] The abscissa of B is the distance between the post and the walker. So, D is a point on the vertical line through B. The point C is the far end of the shadow. So, the distance between the projection of B on the x-axis and point C is the measure of the shadow, that is, it is the ordinate of D. Use a compass to find the point D. [The diagram also shows how points E and F were located.]

[Students who have done some work with similar triangles may recognise that, for each (x, y) which satisfies the conditions of the problem, $y/(x + y) = 5/12$, that is, $y = 5x/7$. So, the relation in question is $\{(x, y):\ x > 0 \text{ and } y = 5x/7\}$.]

2. This question, in effect, asks for the first components of that point in the relation whose second component is 5. That is the point (7, 5). [Actually, geometric intuition alone will tell most students that the far end of the 5-foot shadow ought to be 12 feet from the foot of the post. So, the walker is 7 feet from the post. It is rewarding to find that the graph of the relation is consistent with intuition.

3. 14 feet.

Answers for Part ☆I [on page 5-32].



1. The diagram illustrates a helpful graphical technique for locating points in this relation. [The distance between A and B is the sum of the distance between T_1 and B and between B and T_2.] The relation is $\{(x, y): 0 \leq x \leq 14 \text{ and } y = \sqrt{9 + x^2} + \sqrt{16 + (14 - x)^2}\}$.

2. We seek the first component of the ordered pair whose graph is the lowest point. This ordered pair is $(6, \sqrt{45} + \sqrt{80})$. So, the required location of P is 6 yards from F_1. Another way of solving Exercise 2 [without using differential calculus] is to reflect T_2 in the line through F_1 and F_2. Let T_3 be the image of T_2. The sum of the distances from T_1 to P to T_3 is the same as the sum of the distances from T_1 to P to T_2. But, the smallest of these sums is obtained when T_1, P, and T_3 are collinear. [This is a consequence of the fact that the measure of one side of a triangle is less than the sum of the measures of the other two sides. Avoid saying, in seriousness, that the shortest distance between two points is a straight line. A straight line is not a distance!] By similar triangles we find that this minimizing location of P is such that the distance between F_1 and P is 6.

The purpose of these Exploration Exercises is to prepare students for the notions of the <u>domain</u>, <u>range</u>, and <u>field</u> of a relation. In answering (a) for each exercise of Part A, they name the domain of the given relation; in answering (b) they name its range. The union of the domain and range of a relation is its field, and in Part B on page 5-34, students are asked to name the fields of the relations given in the exercises of Part A. The exercises of Part C prepare students for a new kind of generalization sentence--existential generalization sentences. Such sentences appear as set-selectors in the definitions of domain and range on page 5-35. They have also been discussed, along with universal generalization sentences, in the COMMENTARY for page 2-27 in Unit 2.

*

Answers for Part A [on pages 5-33 and 5-34].

1. (a) $\{6, 7, 8\}$ (b) $\{6, 7, 8\}$

2. (a) $\{2, 3, 4, 5\}$ (b) $\{6, 7, 8, 9\}$

3. (a) $\{-2, -1, 0, 1, 2\}$ (b) $\{5\}$

4. (a) the set of real numbers (b) $\{x: 1 \leq x \leq 2\}$

5. (a) $\{4\}$ (b) the set of real numbers

6. (a) the set of real numbers (b) $\{x: x \geq 0\}$

7. (a) $\{x: |x| < 2\}$ (b) $\{x: |x| < 4\}$

8. (a) $\{x: x \geq 2\}$ (b) $\{x: x \leq -1\}$

9. (a) $\{x: |x| \leq 5\}$ (b) $\{x: |x| \leq 5\}$

10. (a) $\{x: |x| \leq 6\}$ (b) $\{x: |x| \leq 6\}$

11. (a) $\{x: |x| \leq 6\}$ (b) $\{x: 0 \leq x \leq 6\}$

12. (a) $\{x: 0 \leq x \leq 6\}$ (b) $\{x: |x| \leq 6\}$

13. (a) $\{x: 0 < |x| < 6\}$ (b) $\{x: 0 < |x| < 6\}$

14. (a) $\{x: |x| \leq 6\}$ (b) $\{x: |x| \leq 6\}$

15. (a) the set of states in the United States

    (b) the set of cities in the United States other than Washington, D.C.

16. (a) the set of states in the United States

    (b) the set of state capitals in the United States

17. (a) the set of people who have uncles [Note that this is not the set of people who are either nephews or nieces: some of these have only aunts!]

    (b) the set of people who are uncles [or: the set of male people who have a nephew or niece]

*

Answers for Part B.

1. $\{6, 7, 8\}$	2. $\{2, 3, 4, 5, 6, 7, 8, 9\}$
3. $\{-2, -1, 0, 1, 2, 5\}$	4. the set of real numbers
5. the set of real numbers	6. the set of real numbers
7. $\{x: \|x\| < 4\}$	8. $\{x: x \leq -1 \text{ or } x \geq 2\}$
9. $\{x: \|x\| \leq 5\}$	10. $\{x: \|x\| \leq 6\}$
11. $\{x: \|x\| \leq 6\}$	12. $\{x: \|x\| \leq 6\}$
13. $\{x: 0 < \|x\| < 6\}$	14. $\{x: \|x\| \leq 6\}$

15. the set of all political organizations in the United States which are either cities or states, other than Washington, D. C.

16. the set of all political organizations in the United States which are either states or state capitals

17. the set of all people who have or are uncles

*

Answers for Part C.

1. T	2. T	3. F	4. F	5. T	6. T	7. F
8. T	9. F	10. T	11. T	12. F	13. T	

Students may, at first, experience some difficulty in distinguishing between the meanings of the sentences in Exercises 8 and 11 [or those in Exercises 9 and 10, or those in Exercises 12 and 13]. Using the existential quantifier '$\exists$' introduced in the COMMENTARY for Unit 2 on TC[2-27]p and, now, in the text, on page 5-35, these exercises can be written:

$$\left.\begin{array}{l} 8.\ \exists_y(\forall_x\ xy = 0) \\ 11.\ \forall_x(\exists_y\ xy = 0) \end{array}\right\} \qquad \left.\begin{array}{l} 9.\ \exists_y(\forall_{x \neq 0}\ xy = 1) \\ 10.\ \forall_{x \neq 0}(\exists_y\ xy = 1) \end{array}\right\} \qquad \left.\begin{array}{l} 12.\ \exists_y(\forall_x\ x + y = 0) \\ 13.\ \forall_x(\exists_y\ x + y = 0) \end{array}\right\}$$

[The parentheses are unnecessary, but they may make it easier to grasp the sense of the sentences.]

A similar pair of sentences is considered on TC[2-27]i. When rewritten they are:

$$\left.\begin{array}{l} (3')\ \exists_y(\forall_x\ x \leq y) \\ (4')\ \forall_x(\exists_y\ x \leq y) \end{array}\right\}$$

The first says that there is a greatest real number [and, so, is false]; the second says that for each real number there is a real number greater than or equal to it [and, so, is true--each number is less than or equal to itself.].

Of each such pair of sentences, the first makes a stronger claim than the second--more precisely, the first implies the second. The first sentence of such a pair asserts that there is a number which bears a certain relation to each number [or, in the case of Exercise 9, to each nonzero number.] The second sentence asserts only that, for each number, there is a number which bears the relation in question to it. So, for example, since the product of each number by 0 is 0, the sentence of Exercise 8 is true--there is a number [0] such that the product of each number by this single number is 0. Consequently, the sentence of Exercise 11 is also true--for each number there is a number [0] such that the product of the first by the second is 0. The sentence of Exercise 9 is false--there is no single number such that the result of multiplying each nonzero number by it is 1. However, the sentence of Exercise 10 is true--for each nonzero number, there is a number [the reciprocal of the first] such that the product of the first by the second is 1. The sentence of Exercise 12 is false--there

is no single number such that the result of adding it to each real number is 0. On the other hand, the sentence of Exercise 13 is true--for each real number, there is a number [the opposite of the first] such that the sum of the first and the second is 0.

Since, for each pair of sentences of the kind being considered here, the second sentence is a consequence of the first, if the first sentence of such a pair is true then the second must be true. And, if the first is false then, still, the second <u>may</u> be true. That, in this case, the second may also be false is shown by the pair:

$$\left.\begin{array}{l} \exists_y (\forall_x\, xy = 1) \\ \forall_x (\exists_y\, xy = 1) \end{array}\right\}$$

*

If you discussed the imagined graph of U as suggested on TC[5-35, 36]a, you might ask students how they could use the graph to tell if Mr. Adams belonged to the range of U. Find Mr. Adams' name in the vertical list and draw a horizontal line through it. If this line passes through a graph of a point in U, he is in the range. Otherwise, he is not. Next, ask students to consider the set of all points in U whose graphs are on this horizontal line. [This could be the empty set.] What can be said about the first components of these points? They are the nieces and nephews of Mr. Adams. To find out if Bill Smith is in the domain of U, draw a vertical line through his name in the horizontal list. If it hits the graph of the relation, Bill Smith has an uncle. The subset of U whose members have graphs on this vertical line are ordered pairs whose second components are the uncles of Bill Smith. Extend the work a bit by asking students to imagine a vertical line being drawn through the name of Alice Smith, Bill's sister. Will her line hit the same points as Bill's? No, but the second components of the points whose graphs her line contains will be the same as those for Bill's line, if we may assume that brothers and sisters have the same uncles. Ask students to consider the lines they would draw to determine if Mr. Adams belonged to the range and to the domain. Of course, these lines cross each other, but do they cross in the graph of a point which belongs to U? Ask if U contains any point with equal components, but avoid hassles over marriage customs!

*

Some excellent references dealing with relations and their properties are

Tarski, Introduction to Logic (New York: Oxford University Press, 1956),

Suppes, Introduction to Logic (New York: Van Nostrand, 1957),

Huntington, The Continuum (New York: Dover, 1957),

Cogan et al., Modern Mathematical Methods and Models, Volume II (Buffalo, New York: Mathematical Association of America, University of Buffalo, 1958), and

Luce, Some Basic Mathematical Concepts (New Haven: School Mathematics Study Group, 1959).

Some pedagogical suggestions concerning the development of the descriptions (1) and (2) of $\mathcal{D}_R$ and $\mathcal{R}_R$ may be in order since they involve a new type of sentence, existential generalizations. We have a relation R among the members of a set S. We wish to know which members of S are involved in the relation, and, in particular which members of S are first components and which are second components of ordered pairs of R. Let us pick a member of S, say x, and ask if x belongs to the domain of R. To answer this question, we search among the members of S. If we find a member of S, say y, such that (x, y) belongs to R, we say:

Yes, $x \in \mathcal{D}_R$ because there is a $y \in S$ such that $(x, y) \in R$.

Similarly, to find the range of R, we start by picking a member of S, say x, and ask if x belongs to $\mathcal{R}_R$. To answer this question, we search among the members of S for a member, say y, such that (y, x) belongs to R. If we are successful in the search, we say:

Yes, $x \in \mathcal{R}_R$ because there is a $y \in S$ such that $(y, x) \in R$.

*

The symbol '$\exists_{y \in S}$' is pronounced as 'there is a y in S such that'. [For a detailed discussion of universal and existential generalizations, see the essay which begins on TC[2-27]e. The "backwards 'E' " reminds one of the first letter of 'exists' in the phrase 'there exists ... '.]

*

In studying the diagram on page 5-36, students should understand that the graphs of $\mathcal{D}_R$ and $\mathcal{R}_R$ are graphs of subsets of S, and that the graph of R is a graph of a set of ordered pairs whose first components belong to $\mathcal{D}_R$ and whose second components belong to $\mathcal{R}_R$. The graphs of $\mathcal{D}_R$ and $\mathcal{R}_R$ are not graphs of axes. [See page 5-62ff.]

*

Answers for questions in the text [on page 5-36].

Bill Smith is in $\mathcal{D}_U$ if and only if the sentence: (Bill Smith, x) $\in$ U has at least one solution. Mr. Adams is in $\mathcal{D}_U$ if and only if there is at least one solution of: (Mr. Adams, x) $\in$ U

whose domain is P, the set of all people. Similarly, if M is the set of all male human beings, the sentence:

$$\forall_{x \in M} (\exists_{y \in P} \text{ x is an uncle of y})$$

asserts that each male human being is an uncle of some person. Here, '$\in$ M' indicates that 'x' is a variable whose domain is M, and '$\in$ P' indicates that 'y' is a variable whose domain is P. When the domain of a variable is not so indicated it is to be understood that the domain is the set of real numbers [or, if a restriction is attached, a subset of this set]. [Sometimes, as in section 5.02, we may adopt, for a time, some other convention.]

Now, although, as has just been pointed out, questions concerning the domain of a variable are really questions about the use of language, questions concerning the domain [and range] of a relation are questions about the subject matter. Thus, the domain of the <u>relation</u>

$$\{(x, y) \in P \times P\colon \text{ y is an uncle of x}\}$$

is the set of all people who have uncles. This is a proper subset of the domain, P, of the <u>variable</u> 'x'.

*

Note the script letters for domain and range. We want abbreviations for 'the domain of R' and 'the range of R'. ['$\mathcal{D}_R$' is read as 'the domain of R' and '$\mathcal{R}_R$' is read as 'the range of R'.] The script letters and the subscripts serve this purpose. Students should be cautioned about the importance of using script letters instead of Roman letters in these cases. Naturally, they need not make copies of the particular script letters we use in the text. All they need do is to make letters which are clearly distinguishable from the block upper case letters. For example:

$$\mathcal{D} \quad \mathcal{D} \quad \mathcal{R} \quad \mathcal{R}$$

In the case of the field of a relation [page 5-37], letters like:

$$\mathcal{F} \quad \mathcal{F}$$

will do. Students will get practice in making these letters when they do the exercises in Part A on page 5-37.

*

It is important to distinguish between the two uses of the word 'domain'. Students have learned that the domain of a variable is the set of entities which can serve as values of the variable, that is, the set of entities whose names can be used to replace the variable in an expression or in a sentence in which the variable occurs. When one builds a language for the purpose of talking about a particular subject matter, he usually specifies at the outset which symbols he will use as variables and what the domains of these variables are. To ask about the domain of a variable is to ask about one of the ground rules used in setting up the language. In Units 1 through 4, one of our ground rules is that the domain of each variable is a set of real numbers and, unless otherwise specified, is the set of [all] real numbers. Ways of indicating a restriction of the domain are illustrated by:

$\forall_{x \neq 0}\ \frac{0}{x} = 0$	TC[2-84]a
$\{x \neq 0:\ \frac{xx}{x} = \frac{x}{x}\}$	TC[3-27]c
$3x \cdot x = 21 \cdot x,\ [x \neq 0]$	TC[3-45, 46, 47]a
$\{(x, y),\ x \text{ and } y \text{ integers}:\ x = y - 1\}$	4-9
$\{x \in D:\ (x, 3) \in T\}$	TC[5-H]a
$\{(x, y) \in S \times S:\ y < x\}$	5-3

In Unit 5 we have a variety of subject matters--real numbers, people, sets, geometric figures, measures, relations .... One way of making clear what is the subject matter of any particular discussion is to choose, in each case, special symbols as variables. For example, we might decide, once and for all, to use upper case Roman letters as variables whose domain is the set of all people, lower case Greek letters as variables whose domain is the set of all geometric figures, etc. However, this is typographically impractical, and even if it were not, would hardly be worth the trouble. Instead, when we need variables whose domain is not the set of real numbers we shall, as in Exercises 7(f) through 7(p) of Part A on pages 5-8 and 5-9, assign an arbitrary name, say 'D', to the domain and use phrases like '$\in$ D' to indicate that a letter is being used as a variable with this domain. Thus in:

$$\{(x, y) \in P \times P:\ y \text{ is an uncle of } x\},$$

'$\in P \times P$' indicates that both 'x' and 'y' are here being used as variables

The relation U may seem a bit strange since the relations worked with in detail thus far have been numerical ones. It may help to ask students what the ordered pairs are which belong to U. For one thing, they are ordered pairs of people. The first component is a person, and so is the second component. Since these are ordered pairs which belong to the relation of being an uncle of [or: unclehood], the second component of each such ordered pair is an uncle, and the first component is one of his nieces or nephews.

Students may wonder if you can graph the relation U. The answer is 'yes', although, practically speaking, it is impossible. However, it is instructive to discuss the steps you would follow in making such a graph. First, you might make a picture of the cartesian square of the set of all people. One way to do this is to make two lists--one vertical and the other horizontal--of all the people in the world. Then, just as one graphs an ordered pair of numbers, you could graph an ordered pair of people. If you wanted to graph the ordered pair

(Al Brown, Stan Moore),

you would look for Al Brown's name in the horizontal list and draw a vertical line through it; then, look for Stan Moore's name in the vertical list and draw a horizontal line through it. The dot in which the two lines cross is the graph of (Al Brown, Stan Moore). There is also a graph of (Stan Moore, Al Brown) which is different from the graph of (Al Brown, Stan Moore) [assuming Al Brown $\neq$ Stan Moore].

Then, to graph the relation U you would have to select from the cartesian square of P and mark on the picture just those ordered pairs for which the second component is an uncle of the first component.

A brace-notation name for U is: $\{(x, y) \in P \times P:$ y is an uncle of x$\}$

*

Answers for questions in the text.

Each person who is not an uncle [in particular, each female person] is an example of a member of P who is not the second component of any member of U.

A nephew who is so only by virtue of having an aunt is not the first component of any member of U.

The domain of U is the set of people who have uncles.

The range of U is the set of people who are uncles [that is, the set of male persons who have nephews or nieces].

Skill Quiz.

A. Simplify.

1. $\dfrac{\frac{3}{4}}{2}$ 2. $\dfrac{\frac{3}{4} + \frac{1}{2}}{\frac{3}{7}}$ 3. $\dfrac{\frac{1}{5}}{\frac{1}{8} - \frac{2}{9}}$ 4. $\dfrac{\frac{5}{3}}{\frac{8}{9} + 3}$ 5. $\dfrac{\frac{4}{7} - 1}{\frac{4}{7} + 1}$

B. Simplify.

1. $3(2a - b) + (b - a) - 2(6 + c)$ 2. $x(x - y) + y(x - y) - 7(xy) + y^2$
3. $4(x - y)(x + y) - 7(x - y)(x - y) + 6(x - y)^2$
4. $2ab(a + c) - 2bc(a + b)$ 5. $10\left(\frac{x - y}{5}\right) + (x + y)(2 + 3y)$

C. Factor.

1. $x^2 + 5x - 14$ 2. $x^2 - 1$ 3. $3x^2 - 24x + 45$
4. $x^2 + 4xy + 4y^2$ 5. $5 - x^2 + 4x$

D. Solve. [In the case of inequations, give the solution set, using the simplest sentence possible as set selector.]

1. $8x - 14 + 3x = 7$ 2. $5x + 16 < 2x + 9$
3. $x^2 + x = 12$ 4. $8y + 5 < 6y - 7$
5. $x^2 + 4 = 4x$

Answers for Quiz.

A. 1. $\frac{3}{8}$ 2. $\frac{35}{12}$ 3. $-\frac{72}{35}$ 4. $\frac{3}{7}$ 5. $-\frac{3}{11}$

B. 1. $5a - 2b - 2c - 12$ 2. $x^2 - 7xy$ 3. $3x^2 + 2xy - 5y^2$
4. $2a^2b - 2b^2c$ 5. $4x + 3xy + 3y^2$

C. 1. $(x + 7)(x - 2)$ 2. $(x - 1)(x + 1)$ 3. $3(x - 5)(x - 3)$
4. $(x + 2y)(x + 2y)$ 5. $(5 - x)(1 + x)$

D. 1. $\frac{21}{11}$ 2. $\{x: x < -\frac{7}{3}\}$ 3. $-4, 3$
4. $\{y: y < -6\}$ 5. 2

member of the range because adding 1 has an inverse.

[One <u>can</u> solve Sample 2 by saying that

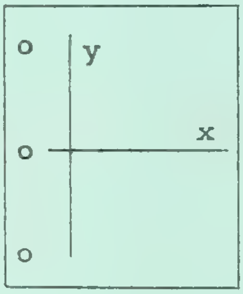
$$\mathfrak{D}_B = \{x\colon \exists_y\, y^2 = x + 1\},$$

and that

$$\mathfrak{R}_B = \{x\colon \exists_y\, x^2 = y + 1\}.$$

But, doing this is akin to answering: $\{x\colon x^2 - 5x + 6 = 0\}$, when asked for the solution set of '$x^2 - 5x + 6 = 0$'!]

A foreknowledge of the domain and range of a relation is useful in drawing a graph. Such knowledge helps you select the scales for the axes. If you were graphing the relation in Sample 2 on a sheet on cross-section paper, you might draw the axes like this:



*

Answers for Part A [on pages 5-37 and 5-38].

1. $\mathfrak{D}_M = \{-2, 0, 5\}$, $\mathfrak{R}_M = \{-2, 0, 1, 5\} = \mathfrak{F}_M$

2. $\mathfrak{D}_N = \{3, 4\} = \mathfrak{R}_N = \mathfrak{F}_N$

3. $\mathfrak{D}_T = \{x \in I\colon 1 \leq x \leq 10\} = \mathfrak{R}_T = \mathfrak{F}_T$

The notion of the field of a relation makes it easy for us to give a precise description of the property of reflexiveness. The statement that the field of R is the smallest subset of S whose cartesian square contains all the members of R may be a little hard for students to understand without having had the experience of doing the exercises in Part A. Don't push this statement, but instead raise these questions which students should think about as they do Part A:

(1) Is each member of $\mathfrak{D}_R$ a member of $\mathfrak{F}_R$? [Yes.]

(2) Is each member of $\mathfrak{R}_R$ a member of $\mathfrak{F}_R$? [Yes.]

(3) Is each member of $\mathfrak{F}_R$ a member of $\mathfrak{D}_R$? [Not necessarily.]

(4) Is each member of $\mathfrak{F}_R$ a member of $\mathfrak{R}_R$? [Not necessarily.]

(5) Can $\mathfrak{D}_R = \mathfrak{F}_R$? [Yes, provided that $\mathfrak{R}_R \subseteq \mathfrak{D}_R$.]

(6) If either $\mathfrak{D}_R$ or $\mathfrak{R}_R$ is S, does $\mathfrak{F}_R = S$? [Yes.]

Follow through by asking for answers when they have completed Part A.

A person who has an obsession both for drawing graphs of relations on square charts and for using as small an area as practicable has probably discovered the notion of a field. He knows that the domain and range of a relation are subsets of the field. So, he prepares his chart to accommodate the cartesian square of the field. This assures him that there will be no point of the relation whose graph will not fit on the chart.

*

Here are pedagogical suggestions for handling Sample 2. The relation B is a relation among the real numbers. We want to find out what real numbers belong to $\mathfrak{D}_B$. So, we pick some real number, say 7, and ask if $7 \in \mathfrak{D}_B$. This is equivalent to asking:

Is there a real number y such that $y^2 = 7 + 1$?

The answer to this question is 'yes' because $\sqrt{8}$ is such a real number. [$-\sqrt{8}$ is another real number whose square is 7 + 1, but the fact that there are <u>two</u> is irrelevant.] Does $-5 \in \mathfrak{D}_B$? That is:

Is there a real number y such that $y^2 = -5 + 1$?

The answer is 'no' because the square of each real number is non-negative. The smallest number in the domain is -1.

Does $12 \in \mathfrak{R}_B$? That is:

Is there a real number x such that $12^2 = x + 1$?

Yes, 143 is such a real number. In fact, each real number is a

4. $\mathfrak{D}_R = \{7\}$, $\mathcal{R}_R = \{2, 6, 7, 9\} = \mathfrak{F}_R$

5. $\mathfrak{D}_S = \{2, 4, 6, 8, 12\}$, $\mathcal{R}_S = \{1\}$, $\mathfrak{F}_S = \{1, 2, 4, 6, 8, 12\}$

6. $\mathfrak{D}_C = \{x\colon\ x \geq 3/2\}$, $\mathcal{R}_C$ = the set of all real numbers = $\mathfrak{F}_C$

7. $\mathfrak{D}_D = \{x\colon\ -5 \leq x \leq 5\} = \mathcal{R}_D = \mathfrak{F}_D$

8. $\mathfrak{D}_E = \{x \in I\colon\ -10 \leq x \leq 10\} = \mathcal{R}_E = \mathfrak{F}_E$

9. $\mathfrak{D}_F$ = the set of all real numbers = $\mathfrak{F}_F$, $\mathcal{R}_F = \{x\colon\ x \geq 4\}$

10. $\mathfrak{D}_G = \emptyset = \mathcal{R}_G = \mathfrak{F}_G$

11. $\mathfrak{D}_H$ = the set of all real numbers = $\mathcal{R}_H = \mathfrak{F}_H$

12. $\mathfrak{D}_J = \{x\colon\ |x| \geq 5\}$, $\mathcal{R}_J$ = the set of all real numbers = $\mathfrak{F}_J$

13. $\mathfrak{D}_K = \{x\colon\ |x| \leq 5\} = \mathfrak{F}_K$, $\mathcal{R}_K = \{x\colon\ |x| \leq 3\}$

Have students make graphs.

*

See Exercise 3 of Part E, Supplementary Exercises [page 5-242], for possible classroom discussion questions at this point. Also, see TC[5-37]a.

*

Answers for Part B.

1. True $[3(11/3) + 7 = 18]$

2. False $[\{x\colon\ x = x + 1\} = \emptyset]$

3. True $[(-1)^2 - 1 = 0]$

4. True $[3 + 8 = 8 + 3]$

5. False [Note, however, that the conjunction '$\exists_x\ 2x - 5 = 0$ and $\exists_x\ 5x - 2 = 0$' is true.] Also, '$\exists_x\ (2x - 5 = -7$ and $5x - 2 = -7)$' is true.]

6. True ['$2(5/2) - 5 = 0$ or $5(5/2) - 2 = 0$' is true.]

7. False $[2 \cdot 1 - 3 \neq 12]$

8. True $[2(15/2) - 3 = 12]$

9. True $[12 = 3 \cdot 4$, and $4 \in I]$

10. False $[\{x\colon\ 12 = 7x\} = \{12/7\}$, and $12/7 \notin I]$

11. True $[0^5 - 3 \cdot 0^4 = 0(0 - 1)(0 + 9)]$

12. True $[\forall_x\ x + (3 - x) = 3]$

13. False $[\forall_y \exists_x\ x + y \neq 3]$

[Note the difference between Exercises 12 and 13. Exercise 12 says that, for each first number there is a second number whose sum with the first is 3; Exercise 13 says that, there is a number such that, no matter what number you add it to, the sum is 3.]

14. True $[\forall_x\ x0 = 0]$

15. True $[\forall_x\ x0 = 0]$

[For a discussion of sentences like those in Exercises 12 through 15, see TC[5-34]b and c.]

*

Answers for Part C [on page 5-38].

[$P \cup Q$ and $P \cap Q$ are relations because the union and intersection of sets of ordered pairs are sets of ordered pairs.]

$$\mathcal{D}_{P \cup Q} = \mathcal{D}_P \cup \mathcal{D}_Q .$$

$\mathcal{D}_{P \cap Q} \subseteq \mathcal{D}_P \cap \mathcal{D}_Q$. However, $\mathcal{D}_{P \cap Q}$ may be a proper subset of $\mathcal{D}_P \cap \mathcal{D}_Q$. For example, consider the relations in Exercises 2 and 3 of Part A. $N \cap T$ is $\{(3, 4)\}$, so $\mathcal{D}_{N \cap T}$ is $\{3\}$. But, $\mathcal{D}_N \cap \mathcal{D}_T$ is $\{3, 4\}$. [Similar remarks apply to ranges and fields.]

*

It is interesting to note that if $\mathcal{D}_R \subseteq S$ and $\mathcal{R}_R \subseteq T$ then, although the domain and range of $\tilde{R}$ (with respect to $S \times T$) are subsets of S and T, respectively, nothing more can be said about $\mathcal{D}_{\tilde{R}}$ and $\mathcal{R}_{\tilde{R}}$ unless one has additional information about R. In particular, it is generally not the case that $\mathcal{D}_{\tilde{R}} = \widetilde{\mathcal{D}_R}$ or that $\mathcal{R}_{\tilde{R}} = \widetilde{\mathcal{R}_R}$.

Quiz.

Suppose that

$R = \{(2, 3), (3, 3), (7, 3), (-2, 3), (0, 3), (-8, 3)\}$,

$T = \{(3, 0), (3, 5), (3, -8), (3, 7), (3, -10), (3, 2), (3, 3), (3, -2)\}$,

$S = \{(0, 5, -8, 7, -10, 2, 3, -2\}$,

$U = \{2, 3, 7, -2, 0, -8\}$,

and $V = \{3\}$.

Complete each of the following to a true sentence with one of the letters 'R', 'T', 'S', 'U', or 'V'.

1. The domain of R is _____.
2. The domain of T is _____.
3. The range of R is _____.
4. The range of T is _____.
5. The field of R is _____.
6. The field of T is _____.
7. The domain of the converse of R is _____.
8. The range of the converse of R is _____.
9. The field of the converse of R is _____.
10. The domain of the converse of T is _____.
11. The range of the converse of T is _____.
12. The field of the converse of T is _____.

*

Answers for Quiz.

1. U	2. V	3. V	4. S	5. U	6. S
7. V	8. U	9. U	10. S	11. V	12. S

4. (a) $\{(2, 6), (9, 9), (3, 8)\}$

(b) $\{(x, y): x = 3y\}$ [or: $\{(x, y): y = \frac{1}{3}x\}$] [Evoke both answers.]

(c) $\{(x, y): x = y\}$

(d) $\{(x, y): y = 3x\}$ [or: $\{(x, y): x = \frac{1}{3}y\}$]

(e) $\{(x, y): y^2 + x^2 = 25\}$

(f) $\{(x, y): y = 2\}$

(g) the relation of being a parent of

(h) the relation of being a cousin of

(i) the relation $<$ [or: $\{(x, y): y < x\}$]

5. $\mathcal{D}_S = K$, $\mathcal{R}_S = J$, $\mathfrak{F}_S = J \cup K$ [Ask the class 'What relation is the converse of S?' The answer is, of course, 'R'.]

6. The converse of a relation R is the relation whose members are obtained by reversing the order of the components of the members of R. [More briefly (and less confusingly): the converse of R is $\{(x, y) \in \mathcal{R}_R \times \mathcal{D}_R : y\ R\ x\}$.]

*

In some textbooks, the converse of a relation R is denoted by '$\breve{R}$'.

*

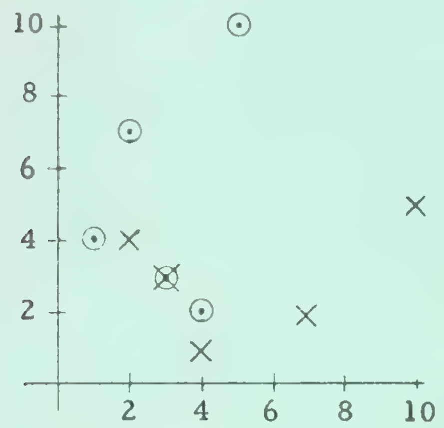
In speaking of the converse of a relation one is referring to the converse of a <u>set</u> of ordered pairs. The word 'converse' is also used in speaking of individual ordered pairs: for each a and b, the converse of (a, b) is (b, a). So, one may say that the converse of a relation R is the relation whose members are the converses of the members of R.

*

Correction. In Exercise 5, delete '(a)'.

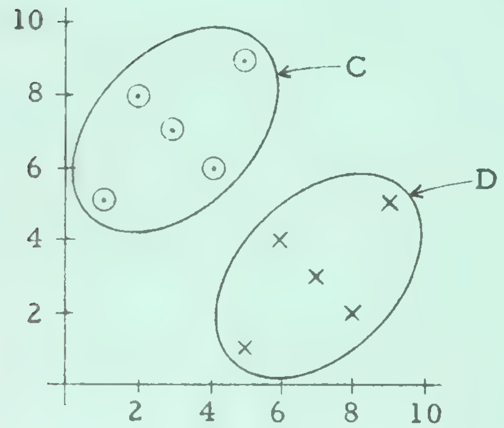
Answers for Part D.

1. (a)

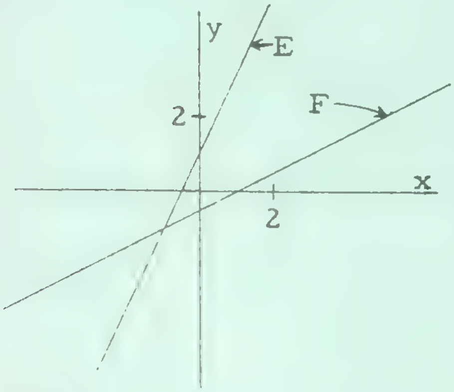


(b) The members of B are obtained by reversing the order of the components of the members of A.

2.



3.



$E = \{(x, y): y - 2x = 1\}$

$F = \{(x, y): x - 2y = 1\}$

Ask students for a brace-notation name for F.

The remainder of this section deals with two of the more important properties which a relation may have. Besides furnishing a good background for the study of functions, it provides an opportunity to bring up interesting combinatorial problems. [Parts D and E on pages 5-43 and 5-44, and Part B on page 5-46].

*

The relation whose graph appears in the middle of page 5-40 is reflexive. Its field is $\{1, 2, 3, 4, 5, 6, 8\}$, and each of the ordered pairs (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), and (8, 8) belongs to the relation.

*

Answers for Part A.

The field of the relation whose graph is (1) is $\{x \in I: 1 \leq x \leq 9\}$. To obtain a reflexive relation one must adjoin to the given relation the pairs (2, 2), (4, 4), (6, 6), (7, 7), and (9, 9). [In addition to these one may adjoin others, for example, (2, 6), or (John, 7). But, if one does include, say, (John, 7) then the relation so obtained has John in its field and, to obtain a reflexive relation, one must then adjoin (John, John).]

The field of the relation whose graph is (2) is $\{x \in I: 2 \leq x \leq 9\}$. To obtain a reflexive relation one must adjoin to the given relation the pairs (6, 6) and (9, 9).

*

After doing Exercise 1, students may jump to the conclusion that a reflexive relation is one whose graph includes "the diagonal". That this is not necessarily the case is seen in Exercise 2, and in the chart in the middle of the page. The point (1, 1) does not have to belong to the relation in order that it be reflexive. It does only if 1 is in the field. Suppose you are preparing to graph a relation which is reflexive, and you decide to make a chart which will just accommodate the graph, that is, it will have no unnecessary columns or rows. The chart will then picture a cartesian square, and the relation is a subset of this square. Furthermore, if the rows are named "going up" in the same order that the columns are named "going to the right" then the graph of the relation will contain all the diagonal points. Moreover, any relation which meets these conditions is a reflexive one.

Exercise 15: Each ordered pair of real numbers with equal components satisfies the set selector because $0^2 + 0^4 = 0$. [Strange as it may look, the relation in Exercise 15 is the same as that in Exercise 3. Since squares and fourth powers of real numbers are nonnegative and since the sum is 0, each addend must be 0.]

*

In later exercises [for example, those in Part D on page 5-43] questions may arise concerning the empty set. For example; Is $\emptyset$ a relation? If one recalls that each set all of whose members are ordered pairs is a relation, he sees that the answer to this question is 'yes'. For, one who claims that $\emptyset$ is <u>not</u> a relation must be prepared to exhibit a member of $\emptyset$ which is not an ordered pair. Since $\emptyset$ has no members, it is impossible that he should be able to do this. Note that

$$\mathfrak{D}_{\emptyset} = \{x\colon \exists_y\ (x, y) \in \emptyset\} = \emptyset \quad \text{and} \quad \mathfrak{R}_{\emptyset} = \{x\colon \exists_y\ (y, x) \in \emptyset\} = \emptyset.$$

Also, $\emptyset$ is reflexive. For, $\mathfrak{D}_{\emptyset} = \emptyset$ and $\forall_{x \in \emptyset}\ (x, x) \in \emptyset$. [Here, again, one who claims that $\emptyset$ is not reflexive faces the impossible task of exhibiting a member of $\emptyset$--this time, an $x \in \emptyset$ such that $(x, x) \notin \emptyset$.]

Finally, $\emptyset$ is [see page 5-45] symmetric. For, there is no ordered pair (x, y) in $\emptyset$ such that $(y, x) \notin \emptyset$.

Answers for Part B.

1. (1, 1), (2, 2), (3, 3), (4, 4), (5, 5) 2. $\mathfrak{R}_R = \{3, 4, 5\}$

3. If R is a reflexive relation then $\mathfrak{D}_R = \mathfrak{R}_R$. [If R is reflexive and $x \in \mathfrak{F}_R$ then, since $x \; R \; x$, $x \in \mathfrak{D}_R$. So, $\mathfrak{F}_R \subseteq \mathfrak{D}_R$. Since, in any case, $\mathfrak{D}_R \subseteq \mathfrak{F}_R$, it follows that if R is reflexive then $\mathfrak{D}_R = \mathfrak{F}_R$. Similarly, if R is reflexive then $\mathfrak{R}_R = \mathfrak{F}_R$.]

4. Yes. [A relation and its converse have the same field.]

*

Answers for Part C.

The relations in Exercises 1, 3, 4, 5, 7, 8, 9, 10, 11, 13, 14, and 15 are reflexive; the others are not.

Exercises 3, 4, 5, 6: Draw the graphs.

Exercise 7: Since $1 \in I^+$, each member of I^+ is a factor of itself. [Suppose G is the set of positive integers greater than 1. Is $\{(x, y) \in G \times G$: y is a factor of x with respect to G$\}$ a reflexive relation? Answer: No. Note that the domain of this relation is the set of composite positive integers.]

Exercise 8: The members of this relation are all the ordered pairs for which the difference of the first component from the second is an integral multiple of 5. For each k in the field of this relation, (k, k) belongs to the relation because $k - k = 0$ and 0 is an integral multiple of 5. [It may help students if they first attempt to graph this relation.]

Exercise 9: Each person has the same parents as himself.

Exercise 12: One is not one's own sister.

Exercise 13: Notice that to name an ordered pair belonging to this relation one would write, for example:

$$(\text{`}3x + 5 = 4 - 7x + 1\text{'}, \text{`}z^2 + 6 = 8 - 2\text{'})$$

That is, one must use <u>names</u> of the components in naming the pair.

So, all together, there are $2^{12} + 4 \cdot 2^6 + 6 \cdot 2^2 + 4 \cdot 2^0 + 2^0$, or 4381, reflexive relations among the members of a given set of four elements.]

Students may be surprised that only about one fifteenth of the relations among a given set of four members are reflexive.

[You may recognize the numbers 1, 4, 6, 4, 1 as the successive coefficients in the expansion of, say, '$(a + b)^4$'. They are sometimes designated by '$\binom{4}{0}$', '$\binom{4}{1}$', '$\binom{4}{2}$', '$\binom{4}{3}$', and '$\binom{4}{4}$', and are called 'binomial coefficients'. So, the number of reflexive relations among the members of a given set of 4 elements is

$$\binom{4}{0}2^{0^2 - 0} + \binom{4}{1}2^{1^2 - 1} + \binom{4}{2}2^{2^2 - 2} + \binom{4}{3}2^{3^2 - 3} + \binom{4}{4}2^{4^2 - 4}$$

or, for short,

$$\sum_{k=0}^{4}\binom{4}{k}2^{k^2 - k} \quad [\text{'}\textstyle\sum\text{' for 'sum'}].$$

For each n, the number of reflexive relations among the members of a given set of n elements is

$$\sum_{k=0}^{n}\binom{n}{k}2^{k^2 - k}.]$$

☆7. 50,625 [The second components of those members of a relation R which have a given member of $\mathfrak{D}_R$ as first component form a nonempty subset of $\mathfrak{R}_R$. If $\mathfrak{R}_R$ is to be a subset of a given 4-member set, then there are $2^4 - 1$ nonempty subsets to choose from. If the domain is a given 4-member set then we must make four such choices in succession. So, the number of relations whose domain is a given 4-member set and whose range is a subset of this set is $(2^4 - 1)^4$. In fact, for each m, for each n, the number of relations whose domain is a given n-member set and whose range is a subset of a given m-member set is $(2^m - 1)^n$.]

Correction. In the third line of Exercise ☆6, change 'Exercise 4' to 'Exercise 5'.

Answers for Part D.

1. 16; 32; 2^{25} [or: 33, 554, 432]

2. 2^{16} [Answer for Hint: 16]

[For each n there are 2^{n^2} relations among the members of a given set of n elements.]

3. 2^{14}

4. $4 \leq n(R) \leq 16$

5. 2^{12} [Both the domain and the range of such a relation is a set of four elements. Each relation with this domain and range is a subset of a 16-member cartesian square. If such a relation is reflexive then it is the union of two sets, a first set consisting of the 4 "diagonal" ordered pairs, and a second set whose members (if any) are among the remaining 12 pairs. So, the question in Exercise 5 amounts to asking how many subsets a set of 12 elements has.]

[For each n, there are $2^{n^2 - n}$ reflexive relations whose field is a given set of n elements.]

☆6. There are 4381 reflexive relations among the members of a given set of four elements. [We begin by classifying the relations in question according to their fields. Then, compute the number of reflexive relations which have a given field, and finally, add the results. There are as many possible fields as there are subsets of the given set of 4 elements. These subsets are (1) the set itself, (2) 4 subsets each of which has 3 members, (3) 6 subsets each of which has 2 members, (4) 4 subsets each of which has 1 member, and (5) the empty set. We have seen in Exercise 5 that there are

$2^{4^2 - 4}$ reflexive relations which have a given 4-element set as field,

$2^{3^2 - 3}$ reflexive relations which have a given 3-element set as field,

$2^{2^2 - 2}$ reflexive relations which have a given 2-element set as field,

$2^{1^2 - 1}$ reflexive relations which have a given 1-element set as field,

and $2^{0^2 - 0}$ reflexive relations which have $\emptyset$ as field.

Answers for Part E [which begins on page 5-43].

1.

n \ k	0	1	2	3	4	5	6	7	8
0	1	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0
2	1	2	1	0	0	0	0	0	0
3	1	3	3	1	0	0	0	0	0
4	1	4	6	4	1	0	0	0	0
5	1	5	10	10	5	1	0	0	0
6	1	6	15	20	15	6	1	0	0
7	1	7	21	35	35	21	7	1	0
8	1	8	28	56	70	56	28	8	1

2.

*

In completing the table, students will be computing the binomial coefficients $\binom{n}{k}$ previously referred to on TC[5-43]b. There are a number of discoveries which they may make while doing so. For example, they should note the symmetry due to the fact that, for each n, for each $k \leq n$, there is the same number of k-member subsets of an n-member set as there are (n - k)-member subsets--that is, $\binom{n}{k} = \binom{n}{n-k}$. Also by taking ratios of corresponding numbers listed in successive rows they <u>may</u> discover that $\binom{n-1}{k} / \binom{n}{k} = \frac{n-k}{n}$, for $0 \leq k \leq n$. For example [n = 6], the ratios of numbers listed in the 5-row to those listed in the 6-row are

1/1, 5/6, 10/15, 10/20, 5/15, 1/6, 0/1,

that is, 6/6, 5/6, 4/6, 3/6, 2/6, 1/6, 0/6.

However, the discovery referred to in Exercise 2 is that,

(1) for each n, $\binom{n}{0} = 1$, and

(2) for each $n \geq 1$, for each $k \geq 1$, $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

So, for example [n = 7], the numbers listed in the 7-row are

1, 1 + 6, 6 + 15, 15 + 20, 20 + 15, 15 + 6, 6 + 1, 1 + 0, 0 + 0.

Those who discover this will want to know why it works. Here is an explanation:

For each n, $\binom{n}{0} = 1$ because each set has just one 0-member subset, the empty set.

Now suppose that, for a given $n \geq 1$ and a given $k \geq 1$ we want to count the k-member subsets of an n-member set S. We can do this by choosing one member, say e_1, of S and counting, first, the number of k-member subsets of S which contain e_1 and, second, the number of k-member subsets of S which do not contain e_1. Now, the k-member subsets of S which contain e_1 correspond exactly with the (k - 1)-member subsets of the (n - 1)-member set $\widetilde{\{e_1\}}$. So, the number of these is the number of (k - 1)-member subsets of an (n - 1)-member set--$\binom{n-1}{k-1}$. And the k-member subsets of S which do not contain e_1 are precisely the k-member subsets of the (n - 1)-member set $\widetilde{\{e_1\}}$. So, the number of <u>these</u> is the number of k-member subsets of an (n - 1)-member set--$\binom{n-1}{k}$. Hence, altogether, the n-member set S has $\binom{n-1}{n-1} + \binom{n-1}{k}$ k-member subsets.

Students may note that any entry in the table can be obtained mechanically when one has discovered properties (1), (2), and

(3) for each $k \geq 1$, $\binom{0}{k} = 0$

[that is, that the empty set has no nonempty subsets!].

The formula '$\binom{n-1}{k} / \binom{n}{k} = \frac{n-k}{n}$' is, because of property (2), equivalent to:

$$\binom{n-1}{k-1} / \binom{n}{k} = \frac{k}{n},$$

or to:

$$n\binom{n-1}{k-1} = k\binom{n}{k}$$

This last formula can be explained as follows:

> As in the explanation of (2), let S be an n-member set, say, $S = \{e_1, e_2, ---, e_n\}$. We know that there are just $\binom{n-1}{k-1}$ k-member subsets of S which contain e_1, $\binom{n-1}{k-1}$ which contain e_2, ---, and $\binom{n-1}{k-1}$ which contain e_n. If we add these results, getting $n\binom{n-1}{k-1}$, we will have counted each k-member subset of S k times--once for each of its k-members. So, we get $k\binom{n}{k}$.

In the case of any of the relations which are not symmetric, all that a student must do is exhibit an ordered pair (a, b) such that (a, b) belongs to the relation and (b, a) does not. Thus, in Exercise 2, (6, 4) belongs but (4, 6) does not. In Exercise 3, (1, 7) belongs but (7, 1) does not. [In connection with Exercise 3, ask if there is a pair (a, b) such that (a, b) and (b, a) both belong. (−5, −5) is such a pair, and it is the only one.]

Note that the relation in Exercise 4 is the union of $\{(x, y): y = 2x + 5\}$ and $\{(x, y): x = 2y + 5\}$. It is the union of a relation and its converse. Such a union is its own converse, and so it is symmetric. It is instructive to ask if the intersection of the components is symmetric. The answer is 'yes' because the intersection consists just of points on the diagonal [in this case, just the point (−5, −5)]. In general, both the union and the intersection of a relation and its converse are symmetric. [Exercise 6 can be discussed in the same manner.]

Exercise 12 deals with the union of a symmetric relation $[\{(x, y): y + x + 1 = 0\}]$ and a nonsymmetric one $[\{(x, y): y - x + 1 = 0\}]$. In this case, the union is nonsymmetric. But, this is not generally so. For if R is symmetric, T nonsymmetric, and $T \subseteq R$, $R \cup T$ is symmetric.

Exercise 13 provides an example of the intersection of a nonsymmetric relation and its converse. [Recall that '$y - 1 < x < y + 1$' is equivalent to '$y - 1 < x$ and $x < y + 1$'. ['$x < y + 1$' is equivalent to '$x - 1 < y$', which is the "converse" of '$y - 1 < x$'.] And, $\{(x, y): y - 1 < x \text{ and } x < y + 1\} = \{(x, y): y - 1 < x\} \cap \{(x, y): x < y + 1\}$.]

Although students could do all of these exercises by drawing graphs and searching for symmetry with respect to the diagonal [certainly, a few of them should be done this way], we hope that Exercises 9 and 10 will be done merely by interchanging 'x' and 'y' in the set selectors.

*

As an eleventh exercise for Part B, it would be natural to ask how many symmetric relations there are whose field is a given set of five elements. Notice that this question is related to that of Exercise 8 in the same way as that of Exercise 9 is related to that of Exercise 10 [and not as that of Exercise 10 is related to that of Exercise 9]. The solution of such problems requires a more complicated technique based on extensions of the formula:

$$(*) \qquad n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

We cannot go into this here but, for your information, the answer to the proposed exercise is

$$2^{(5^2+5)/2} - 5 \cdot 2^{(4^2+4)/2} + 10 \cdot 2^{(3^2+3)/2} - 10 \cdot 2^{(2^2+2)/2} + 5 \cdot 2^{(1^2+1)/2} - 2^{(0^2+0)/2},$$

or 28217. One subtracts from the number [2^{15}] of symmetric relations among the members of a given 5-member set S the number of such relations whose fields are proper subsets of S. Each of these relations is a symmetric relation among the members of one of the five 4-member subsets of S. So, if A is the set of symmetric relations among the members of one of these sets, and B, C, D, and E are, respectively, the sets of symmetric relations among the members of the other four 4-member subsets, what we need to calculate is $n(A \cup B \cup C \cup D \cup E)$. For this we need an extension of (*). Using it, and the fact that, for example, $n(A) = 2^{10}$, $n(A \cap B)$ = the number of symmetric relations among the members of a 3-member subset of $S = 2^6$, $n(A \cap B \cap C) = 2^3$, etc., one arrives at the result given above.

*

Answers for Part C [on page 5-46].

The relations in Exercises 1, 4, 5, 6, 7, 8, 11, and 13 are symmetric, the others not. [By the time students have done Exercises 4, 5, 6, 7, 8, 11, and 13, they should see that a relation described as these are is symmetric if, when one interchanges 'x' and 'y' in the given set selector, one obtains an equivalent sentence. Exercises 3, 9, 10, 11, and 12 make the point that this condition is necessary (as well as being sufficient) for symmetry.]

8. 2^{15} [This answer can be obtained by a procedure similar to that used in solving Exercise 5 of Part D on page 5-43. Suppose $S = \{1, 2, 3, 4, 5\}$. A relation among the members of S is a subset of the 25-member set $S \times S$ (and there are 2^{25} such subsets). Each symmetric relation among the members of S can be thought of as resulting from choosing members of $\{(x, y) \in S \times S: x \geq y\}$, and then choosing those members of $S \times S$ "above" the diagonal which are required by symmetry. Since $\{(x, y) \in S \times S: x \geq y\}$ has 15 members $[5 + (5^2 - 5)/2 = 15]$, there are 15 choices to be made. A slight variation is to think of the relation as determined by choosing any subset of the diagonal of $S \times S$ and any subset of the members of $S \times S$ below the diagonal. Since $S \times S$ has 5 diagonal members and $(5^2 - 5)/2$ members below the diagonal, there are 2^5 subsets of the diagonal and 2^{10} subsets whose members are below the diagonal. So, the first choice can be made in 2^5 ways and the second in 2^{10} ways, and there are $2^5 \cdot 2^{10}$ combinations of choices. In general, for each n, there are $2^{(n^2 + n)/2}$ symmetric relations among the members of a given set of n elements.]

9. 2^{10} [This exercise differs from Exercise 8 only in that, here, one has no choice with respect to a diagonal member of $S \times S$: all such ordered pairs must be chosen. So, there remain 10 choices, 2^{10} outcomes. For each n, there are $2^{(n^2 - n)/2}$ relations which are both reflexive and symmetric and whose field is a given set of n elements.]

☆10. 1450 [Exercise 10 is related to Exercise 9 in much the same way as Exercise 6 of Part D on page 5-43 is related to the exercise which precedes it. A relation among the members of a set of 5 elements may have as its field either (1) the empty set, (2) one of 5 singleton sets, (3) one of 10 2-member sets, (4) one of 10 3-member sets, (5) one of 5 4-member sets, or (6) the given set itself. Using the result obtained in Exercise 9, one sees that there are, all together,

$$2^{(0^2 - 0)/2} + 5 \cdot 2^{(1^2 - 1)/2} + 10 \cdot 2^{(2^2 - 2)/2} + 10 \cdot 2^{(3^2 - 3)/2}$$
$$+ 5 \cdot 2^{(4^2 - 4)/2} + 2^{(5^2 - 5)/2},$$

or 1450, relations among 5 elements which are both reflexive and symmetric. For each n, there are

$$\sum_{k=0}^{n} \binom{n}{k} 2^{(k^2 - k)/2}$$

relations among n elements which are both reflexive and symmetric.]

Answers for Part A.

(1) (1, 6), (4, 7), (5, 7), (6, 2), (6, 3), (6, 5), (6, 7), (6, 8), (6, 9), (8, 3), (8, 7), (9, 3), (9, 7)

(2) (1, 2), (1, 6), (2, 3), (2, 6), (3, 1), (3, 6), (4, 5), (4, 6), (5, 6), (7, 4), (8, 9), (9, 7)

(3) None.

(4) (2, 7), (2, 9), (4, 5), (4, 7), (6, 3), (6, 5), (8, 1), (8, 3), (10, 1)

*

Answers for Part B [on page 5-46].

1. Yes; yes; yes; yes. [A symmetric relation among the members of set S must contain an even number (possibly 0) of "nondiagonal" pairs in $S \times S$ and may contain any number of pairs from the diagonal of $S \times S$. So, symmetry puts no restriction on the number of members of a relation.]

2. Yes. [Of course.]

3. Yes. [One example is $\{(1, 1), (2, 2), (3, 3)\}$.]

4. $\mathfrak{R}_R = \{3, 4, 5\}$

5. If R is a symmetric relation then $\mathfrak{D}_R = \mathfrak{R}_R$. [If $x \in \mathfrak{D}_R$ then there is a y such that x R y. And, if R is symmetric, y R x. So, $x \in \mathfrak{R}_R$. Hence, $\mathfrak{D}_R \subseteq \mathfrak{R}_R$. Similarly, $\mathfrak{R}_R \subseteq \mathfrak{D}_R$.]
   If R is symmetric then the converse of R is R itself.

6. A relation which is its own converse is symmetric. [So (Exercise 5), a relation is symmetric if and only if it is its own converse.]

7. We know that there are at least five and no more than twenty-five members in R, but we are not sure of any given one. [Since R is symmetric, $\mathfrak{D}_R = \mathfrak{F}_R$. So, each member of $\mathfrak{F}_R$ must be the first component of some member of R. Hence, $5 \leq n(R)$.]

$*$

Quiz.

A. For each relation described below, give its domain, range, and field.

1. $T = \{(-1, 7), (-3, 5), (9, 8), (7, 2), (6, 3), (31, 27)\}$

2. $M = \{(3, 7), (5, 9)\}$

3. $F = \{(x, y): x^2 = 16 - y^2\}$

4. $G = \{(x, y) \in I \times I: |x| < 5 - |y|\}$

5. $K = \{(a, b): 2b = 7a - 15\}$

B. For the relations described below tell whether

(i) the relation is reflexive, (ii) the relation is symmetric.

1. $\{(x, y): x \not< y\}$

2. $\{(x, y): x - y = 0\}$

3. $\{(a, b): |a| - |b| = 0\}$

4. $\{(p, q): p = 5q + 3\}$

5. $\{(x, y): x = y^2 + 2\}$

$*$

Answers for Quiz.

A. 1. $\mathfrak{D}_T = \{-3, -1, 6, 7, 9, 31\}$

$\mathfrak{R}_T = \{2, 3, 5, 7, 8, 27\}$

$\mathfrak{F}_T = \{-3, -1, 2, 3, 5, 6, 7, 8, 9, 27, 31\}$

2. $\mathfrak{D}_M = \{3, 5\}$

$\mathfrak{R}_M = \{7, 9\}$

$\mathfrak{F}_M = \{3, 5, 7, 9\}$

3. $\mathfrak{D}_F = \{x: -4 \leq x \leq 4\} = \mathfrak{R}_F = \mathfrak{F}_F$

4. $\mathfrak{D}_G = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\} = \mathfrak{R}_G = \mathfrak{F}_G$

5. $\mathfrak{D}_K$ = the set of all real numbers $= \mathfrak{R}_K = \mathfrak{F}_K$

B. 1. r 2. r, s 3. r, s 4. neither 5. neither

A little help may be needed for Exercise 5. Students do have some kind of concept of right angle. But, to understand the set selector in Exercise 5, they must think of an angle as the union of two noncollinear rays with a common end point. So, an angle is a set of points. The union of a line and a line is also a set of points. And, we are looking for ordered pairs of lines such that the union of the first and the second contains a right angle as a subset. Since these are lines [not rays], the right angle will be a proper subset of the union. In fact, each union which contains a right angle also contains three other right angles.

*

Answers for Part E [on page 5-47].

[To save space, we omit answers for parts (a) of Exercises 1-12.]

1. (b) yes (c) yes (d) congruence (of triangles)
2. (b) no (c) yes (d) parallelism [We assume that a line is not parallel to itself.]
3. (b) yes (c) yes (d) congruence (of circles)
4. (b) yes (c) yes (d) congruence (of segments)
5. (b) no (c) yes (d) perpendicularity
6. (b) yes (c) yes (d) similarity (of triangles)
7. (b) yes (c) yes (d) congruence (of triangles)
8. (b) yes (c) yes (d) similarity (of triangles)
9. (b) yes (c) yes (d) congruence
10. (b) yes (c) yes (d) similarity
11. (b) yes (c) yes (d) [none]
12. (b) yes (c) yes (d) equivalence (of triangles with respect to area)

*

As in the answer for Exercise 9, we use 'congruence' as an abbreviation for 'congruence of plane geometric figures'. Similarly for 'similarity'. The parenthetical phrases in answers for Exercise 1, 3, 4, 6, 7, and 8 point out that the relations in question are subsets of these "larger" relations.

Although there is no "commonly used name" for the relation in Exercise 11, one might use 'equivalence of triangles with respect to perimeter'.

vertices to see whether one triangle can be made to "fit" the other. After a bit of experimentation, you should be able to tell that a pair of triangles will belong to the relation just if they agree in their angles and their sides with respect to at least one matching of their vertices. [For a more formal discussion of triangle-congruence, see Unit 6.]

[Students should depend upon geometric intuition to make their decision about whether one can decide on the truth of (*) just by looking at the sentence. We hope that most of your students can do this without a lot of drawing or model-making. You should be in a fairly good position to judge their ability in view of what they did in section 5.03. You may want to come to class equipped with a dozen cardboard or wire models of triangles, some of which are similar, some with the same area, some congruent, etc. In any event, a correct answer for part (a) consists of two pictures, each of a triangle, such that the triangles agree in their angles and in their sides.]

The relation is reflexive since each triangle has the same size and shape as itself. The relation is symmetric since if a first triangle has the same size and shape as a second triangle, then the second triangle has the same size and shape as the first. [As you can see, the word 'same' is a key one in responding to parts (b) and (c).]

In answering part (d), much depends upon the geometry studied in earlier courses. Undoubtedly, some of your students will know the words 'congruent' or 'congruence'. This is an opportunity to acquaint all of them with it. In view of the other relations in the list to which the word 'congruence' applies, it is best to call the relation in Exercise 1 'triangle-congruence' or 'congruence of triangles'. [It is rather interesting to note that those students who do not know the word 'congruence' still have an awareness of the relation, an awareness at least partially developed by the exercise itself.]

*

Do not insist too strongly on answers for parts (d); otherwise, there may be considerable head-scratching at home for Exercises 11 and 12.

*

Correction. In the figure on page 5-49, the vertical dotted segment should start at (48, 0) and the horizontal dotted segment should end at [approximately] (0, 50).

Answers for Part D.

1. The relations in Exercises 2 and 13 of Part C are reflexive; the others are not.

2. The only symmetric relations in Part C on page 5-41 are those in Exercises 3, 5, 6, 8, 9, 10, 11, 13, 14, and 15.

*

Before you assign the exercises of Part E, we suggest that you discuss Exercise 1 with the class [or, at least make a start on it]. Ask them to look, first, at the index '$(x, y) \in T \times T$' in the name of the relation. This tells you that the relation in question is a subset of the cartesian square of the set of all triangles: and, this means that the domain of 'x' and of 'y' is the set of all triangles. So, a member of this relation is an ordered pair of triangles. To do part (a) for Exercise 1, then, you will [if the relation is not empty] need to draw pictures of two triangles. But, what kind of triangles?

To answer this question, think about the triangles which belong to this relation. An ordered pair of triangles belongs to the relation if and only if the set selector 'x has the same size and shape as y' is converted into a true sentence by substituting for 'x' a name for the first component and substituting for 'y' a name for the second component. [Notice that we do not put triangles in place of 'x' and 'y' in the set selector; we put names of triangles in those places.] How do we find triangles whose names will convert the set selector into a true sentence?

Let's use a bit of imagination. Pretend that you have access to the set of all triangles [please don't ask us where those triangles are!]. Each triangle has a tag attached to it [please don't ask us how!] and a name of the triangle is printed on the tag. The tags might bear such names as 'ΔJIM', 'ΔLIZ', 'Δ3', 'ΔD', etc. If you choose one, and then another, of those triangles at random, and put a name for the first one chosen in place of 'x' in the set selector, and a name for the second one in place of 'y', can you determine whether the set selector has been converted into a true sentence? You might get a statement like:

(*) ΔJIM has the same size and shape as ΔCAL

Of course, you can not decide whether the sentence is true just by looking at it! You must look at the triangles! You may be able to tell at once that ΔJIM is bigger than ΔCAL. But, the triangles may appear to be about the same size; if so, you will have to try matching the

<u>Correction.</u> In the Sample on page 5-50, insert '(7, 9)' after '(7, 5)'.

When asking whether a given relation is a function, if 'No' is a correct answer, don't accept 'No, it's a relation'. This way of speaking tends to establish the incorrect belief that functions are not relations and that relations are those sets of ordered pairs which are not functions.

*

Answers for Part A [on pages 5-50 and 5-51].

1. Not a function. [It contains two ordered pairs, (6, 5) and (6, −4), which have the same first component.]
2. Function.
3. Not a function. [It contains two ordered pairs, for example, (−1, 6) and (−1, 4), which have the same first component. This relation contains <u>five</u> ordered pairs.]
4. Function.
5. Function. [Since $\sqrt{4} = 2$, '(3, 2)' and '(3, $\sqrt{4}$)' name the same ordered pair.]
6. Function.
7. Function. [Any singleton of ordered pairs is a function. Since there are not two ordered pairs in a singleton, there cannot be two ordered pairs with the same first component.]
8. Function. [An argument like that in Exercise 7 applies here.]
9. Function.
10. Not a function. [It contains two ordered pairs, for example, (0, $\sqrt{5}$) and (0, $-\sqrt{5}$), which have the same first component.]
11. Not a function.
12. Function.
13. Not a function.
14. Not a function.
15. Function.
16. Not a function.

*

Answers for Part B [on page 5-51].

The converses of the relations in Exercises 1, 3, 7, 8, 9, and 13 are functions; the converses of the others are not.

<u>Correction.</u> In Exercise 7, insert a '}' after '$\sqrt{x^2}$'.

IMPORTANT NOTICE

The material on page 5-52 was inadvertently misplaced. Have your class skip page 5-52, and proceed to a study of pages 5-53 through 5-60. After the exercises on page 5-60 have been studied, and students are acquainted with functional notation, you should assign the material on page 5-52.

The purpose of Part C on page 5-52 is to point out that, given functions f and g, in order to show that f = g it is necessary to show, not only that $f(x) = g(x)$ for each $x \in \mathcal{D}_f \cap \mathcal{D}_g$, but also that $\mathcal{D}_f = \mathcal{D}_g$. Your students, for whom functions are sets of ordered pairs, should have no difficulty in seeing that this is the case. [With some other approaches to this notion of function, the pedagogical problem is more troublesome.]

*

Answers for Part C.

1. (a) yes (b) no (c) no 2. (a) yes (b) no (c) no

3. (a) yes (b) no (c) no [A 'no' answer for part (a) requires that one find a common argument of f and g. But, $\mathcal{D}_f \cap \mathcal{D}_g = \emptyset$ (and $f \cap g = \emptyset$).]

4. (a) yes (b) no (c) no [Although $g \neq f$, $g \subseteq f$.]

5. (a) yes (b) yes (c) yes 6. (a) yes (b) no (c) no

7. (a) yes (b) yes (c) yes [Another reminder that $\forall_x \sqrt{x^2} = |x|$.]

*

8. No. For example, both $\{(x, y): y = x\}$ and $\{(x, y): y = x + 1\}$ have the same domain and the same range.

Here is a similar problem which may amuse your students.

> A boy has a drawer which contains 26 blue socks and 26 brown socks. If, without looking, he reaches into the drawer and takes out socks, one at a time, how many must he take out to be sure of having two of the same color?

Many people will give the snap answer: 27 The correct answer, of course, is: 3

☆18. Function. [The domain of 'x' is the set of all rectangles, and the domain of 'y' is the set of all ordered pairs of numbers of arithmetic. If one adopts the convention that the width of a rectangle does not exceed its length, the set selector assigns to each rectangle exactly one ordered pair of dimensions. If one does not adopt some such convention, the relation is not a function.]

9. Function.

10. Function. [In speaking of the area-measure, we assume that some particular unit has previously been specified. A similar remark applies in Exercise 11.]

11. Function.

12. Function.

13. Function. [Each triangle has one and only one circumscribed circle.]

14. Not a function. [Each circle circumscribes many triangles.]

15. Function.

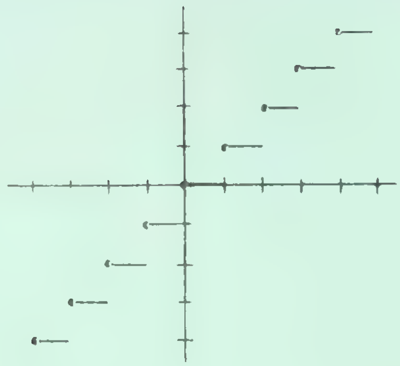
16. [As in Exercises 10 and 11, we assume that units of weight and pressure have been chosen.] (a) Function, (b) Function, (c) Function, (d) Function, (e) Function, (f) The relation is a function if (and only if), for each two days on which Hamster 7's weight is the same, his blood pressure is also the same.

17. (a) Function.

    (b) Not a function. [In addition to the information given in the bracketed sentence you also need to know that there are more than a million and one persons in New York. Hence, if no two of the first 1 000 001 ordered pairs have the same first component, then the 1 000 002nd ordered pair must have the same first component as one of the first 1 000 001. For your information, some authorities claim that no person has more than 140 000 hairs on his head!]

    (c) If there do not exist at least 3 students who have birthdays falling in the same month then at most 2 students have birthdays in each month. If this is the case then, since there are exactly 12 months, the class can have at most 24 students.

6. Function.



7. Function. [This exercise may be difficult for the students. First they must realize that the domain of 'x' is a set of ordered pairs--the set of pairs whose components are the components of points of the unit circle. This set of pairs is $\{(a, b): a^2 + b^2 = 1\}$. So, some of the members of the domain of 'x' are

$$(-1, 0),\ (0, 1),\ (\frac{1}{2}, \frac{\sqrt{3}}{2}),\ (0, -1),\ (1, 0),\ (\frac{1}{2}, -\frac{\sqrt{3}}{2}).$$

After a little thought, students should realize that the domain of $\{(a, b): a^2 + b^2 = 1\}$ is $\{x: -1 \leq x \leq 1\}$. Hence, the first component of a point x must be a number which is equal to or greater than -1 and less than or equal to 1.

Next, students should think about how one finds pairs which belong to the relation. We choose a point x [that is, an ordered pair which is in the domain of 'x']. Suppose we choose (-1. 0). Now, the first component of this point is -1, so, according to the set selector, the pair $((-1, 0), -1)$ belongs to the relation. If we choose another point x, say, the point (1, 0), we get ((1, 0), 1), and this is a pair which is a member of the relation. Still other pairs which belong to the relation are

$$((0, 1), 0),\ ((\frac{1}{2}, \frac{3}{2}), \frac{1}{2}),\ ((0, -1), 0),\ ((\frac{1}{2}, -\frac{\sqrt{3}}{2}), \frac{1}{2}).$$

Be sure students see that, even though the pairs (0, 1) and (0, -1) which are in the domain of 'x' do have the same first component, [C_1 is not a function], they are <u>different</u> pairs. So, when we use them to get pairs which belong to the relation in question [((0, 1), 0) and ((0, -1), 0)] we get ordered pairs with different first components.]

8. Not a function. [For example, (0, (0, 1)) and (0, (0, -1)) both belong to the relation.]

Answers for Part D [on pages 5-53 and 5-54].

[The kind of discussions referred to on TC[5-35, 36]a and TC[5-47]a should pay off in doing these exercises. For example, in Sample 1, students should understand the nature of the domains of 'x' and 'y'. The ordered pairs in this relation are ordered pairs of people. Note that not every member of the domain of 'x' belongs to the domain of the relation. The ordered pairs in the relation in Sample 2 are ordered pairs whose first components are circles and whose second components are points. In this case, the domain of 'x' is the domain of the relation, and the range of the relation is the domain of 'y'.]

1. Not a function. [Since each male has two parents, for each ordered pair in the relation, there is another which has the same first component but a different second component.]

2. Function. [Note that this relation is a subset of the one in Exercise 1. Also, both relations have the same domain.]

3. Function. [In discussing this exercise, you may have some students who will say that the relation is not a function. Their argument will probably be that the relation might contain such pairs as

   (Mr. S. A. Zick, 2), (Mr. I. E. Aye, 1), (Mr. S. A. Zick, 3)

   since, if Mr. Zick owns 3 automobiles it is certainly the case that he owns 2! If this argument arises, point out to the students that, in ordinary usage, when one says 'Mr. Zick owns 3 automobiles', one means that Mr. Zick owns exactly 3 automobiles. This is the convention we are using in Exercise 3 [and, in Exercises 5, 9, 12, 14, and 17(a)]. Students should realize, also, that the range of this function is a subset of the set of whole numbers, and could be determined by analyzing automobile registrations. Certainly, 0, 1, and 2 belong to the range, as do several other small whole numbers. There is surely a largest member of the range. Very likely, not all whole numbers between 0 and this largest number belong to the range... .]

4. Not a function.

5. Function. [Ask students if the converse of this relation is a function. Since it is reasonable to suppose that there were the same number of traffic deaths on some two days, the converse is not a function.]

Quiz.

(a) Which of the relations listed below are functions?

(b) Which of the converses of the relations listed below are functions?

1. $\{(5, 0), (0, 0), (-2, 3), (-5, 0)\}$

2. $\{(0, 5), (0, 0), (-2, 3), (-5, 8)\}$

3. $\{(x, y) \in \text{Reals} \times \text{Numbers of arithmetic}: y = |x|\}$

4. $\{(a, b) \in \text{Numbers of arithmetic} \times \text{Reals}: a = |b|\}$

5. $\{(0, 0)\}$

6. $\{(r, s) \in \text{Nonnegative reals} \times \text{Reals}: s = -r \text{ or } s = r\}$

7. $\{(m, n) \in \text{Reals} \times \text{Reals}: n = m\}$

8. $\{(p, q) \in \text{Reals} \times \text{Reals}: q = p\}$

9. $\{(c, d) \in \text{Reals} \times \text{Reals}: c = d\}$

10. $\{(x, y): y^2 = 25 - x^2\}$

11. $\{(x, y), |x| \leq 5: y = \sqrt{25 - x^2}\}$

*

Answers for Quiz.

(a) 1, 3, 5, 6, 7, 8, 9, and 11 are functions.

(b) The converses of 2, 4, 5, 7, 8, and 9 are functions.

9. $\{(x, y) \in \text{Rats} \times N\text{: } x \text{ learns the maze in } y \text{ trials}\}$

10. $\{(x, y) \in \text{Mazes} \times N\text{: the rat learns } x \text{ in } y \text{ trials}\}$

11. $\{(x, y) \in \text{Nonnegative numbers} \times \text{Nonnegative numbers: } y = \sqrt{x}\}$

12. $\{(x, y) \in \text{Reals} \times \text{Reals: } y = -x\}$

13. $\{(x, y) \in \text{Reals} \times \text{Nonnegative numbers: } y = x^2\}$

14. $\{(x, y) \in \text{Positive integers} \times N\text{: } x \text{ has } y \text{ prime factors}\}$

*

On page 5-56, the parentheses in the expression 'F(3)', rather than serving as grouping symbols, indicate application of a function to an argument. In such contexts, '( )' is the translation into mathematical language of the English word 'of'. 'of' generally indicates function-application. For example, the use of the word 'of' in the next-to-last sentence above indicates application of the function whose arguments are English words and whose values are their translations into mathematical symbols. Using 'T' as a name for this function, the sentence in question can be rewritten: '( )' = T ('of'). So, when you see 'of', look for a function. [For example, the two other uses of 'of' in this paragraph suggest that the word 'application' refers to a function. In fact, application is a "function of two variables" [See page 5-109.] One of its ordered pairs is ((F, 3), 5).] In this light, such symbols as '35%' are seen to be names of functions [See TC[1-59].] 35% = multiplying by 0.35; and 35% of 48 = 35% (48) = 48 × 0.35.

*

Answers for Part E.

1. Ex. 9	2. Ex. 3	3. Ex. 6	4. Ex. 13
5. Ex. 7	6. Ex. 10	7. Ex. 5	8. Ex. 17 (a)
9. Ex. 16 (e)	10. Ex. 15	11. Ex. 12	12. Ex. 16 (c)
13. Ex. 18	14. Ex. 11		

*

Answers for Part F [on pages 5-55 and 5-56].

1. $\{(x, y) \in$ Families $\times N$: there are y children in x$\}$
   [In place of 'N' we could have used a 'W' to refer to the set of whole numbers. A similar remark applies to the index given for Exercises 6, 9, 10, and 14.]

2. $\{(x, y) \in$ Triangles $\times N$: y is the perimeter of x$\}$

3. $\{(x, y) \in$ Circles $\times N$: y is the diameter of x$\}$
   [The word 'diameter' is ambiguous. A diameter of a given circle is a segment whose end points belong to the circle and which contains the center of the circle. The diameter of a given circle is the measure of such a segment.]

4. $\{(x, y) \in$ Males $\times N$: y is the dollar-income of x during his lifetime$\}$

5. $\{(x, y) \in$ Years $\times N$: y is the dollar-income of John Wilson during x$\}$

6. $\{(x, y) \in$ Zabranchburg High classes $\times N$: there are y students in x$\}$
   [See comment for Exercise 1.]

7. $\{(x, y) \in$ Years $\times N$: y inches of rain fell in Portland, Oregon during x$\}$

8. $\{(x, y) \in$ Days $\times$ Reals: y is the temperature of the frog on x$\}$
   [Relative measures, such as centigrade or Fahrenheit temperature-measures, are real numbers. They measure a "directed trip" from the temperature of melting ice [C], or of a melting mixture of ice and salt [F], to the temperature of the observed body. It would be exactly analogous to measure the height of an object as $^-40$ if it were 40 feet below the roof level of a given building, and $^+30$ if it were 30 feet above the roof level. [Measures of absolute temperature are numbers of arithmetic.]]

13. (a) 1 (b) -1 (c) $3/2$ (d) $-3/2$ (e) 0
(f) -481 (g) $\{x: x \neq 0\}$

*

Answers for Part C [on page 5-59].

1. $\{2, 6, 24\}$ 2. $\{0\}$ 3. the set of integers
4. $\emptyset$ 5. $\{9\}$

Answers for Part D [on page 5-60].

1. 6 2. $5\frac{1}{2}$ 3. 2 4. $1\frac{1}{8}$
5. 7 6. 7 7. $2\frac{1}{2}$ 8. $1\frac{1}{2}$

9. No. Since $P(r_3) = P(r_4)$, but $A(r_3) \neq A(r_4)$, the relation contains two ordered pairs $[(7, 2\frac{1}{2})$ and $(7, 1\frac{1}{2})]$ with the same first component.

No. There are two rectangles [not among those pictured] which have the same area-measures, but different perimeters. For example, suppose r_5 has width-measure 3 and length-measure 5, while r_6 has width-measure 2.5 and length-measure 6. Both have area-measure 15, but $P(r_5) = 16$ while $P(r_6) = 17$.

10. (a) 14; 10 (b) 4; 26 (c) 30
(d) 14 (e) 2; 3

7. (a) 4 (b) 5 (c) −5 (d) −5 (e) 0 (f) −1

(g) [$3.7 \notin \mathcal{R}_G$] (h) [Any numeral for a real number in $\{x: 4 \leq x < 5\}$ is a correct answer.]

(i) the set of real numbers (j) the set of integers

[The function G of Exercise 7 is sometimes called 'the greatest integer function'. Its graph is shown on TC[5-53, 54]b.]

8. (a) 1 (b) −1 (c) 1

(d) [Any numeral for a nonnegative real number is a correct answer.]

(e) [$0 \notin \mathcal{R}_f$]

(f) [Any numeral for a negative real number is a correct answer.]

(g) the set of real numbers (h) $\{1, -1\}$

9. (a) 3 (b) 1 (c) Martin (d) Ruth

(e) [George $\notin \mathcal{D}_A$] (f) 8

(g) {Mary, Ruth, Martin, Alice} (h) $\{1, 3, 5, 8\}$

10. (a) 25 (b) −10 (c) 500 (d) 2500 (e) 21

(f) 2×10^4 (g) 0 (h) −1 (i) the set of real numbers

11. (a) [$1 \notin \mathcal{D}_F$] (b) 2 (c) [$2 \notin \mathcal{D}_F$]

(d) [$-2 \notin \mathcal{D}_F$] (e) 30 (f) 8 (g) [$2.5 \notin \mathcal{D}_F$]

(h) 20.25 (i) [$0 \notin \mathcal{R}_F$] (j) $\{x: x > 0\}$

12. (a) 12 (b) 2 (c) 22 (d) 2 (e) 2/3

(f) −2 [or: −1] (g) $\{x: x \geq -1/4\}$

[Students may guess the correct answer to part (g) of Exercise 12 after drawing a graph of the function h. They can obtain a check on the graph by using methods learned in Unit 3 to solve the inequation '$x^2 + 3x + 2 < 0$'. These methods involve factoring '$x^2 + 3x + 2$' and the factored form, '$(x + 2)(x + 1)$', may suggest the equivalent expressions '$[(x + 3/2) + 1/2]\,[(x + 3/2) - 1/2]$' and '$(x + 3/2)^2 - 1/4$'. If so, they can at once verify their guess as to the answer for part (g). This procedure of "completing the square by factoring" is sometimes helpful in studying quadratic functions.]

Answers for Part A.

1. 4   2. 8   3. 4   4. −22

*

Note that part (d) of Sample 1 for Part B cannot be answered by filling the blank. For example, 'g(13) = nonsense' is, itself, nonsense! In the solution given for part (d), the 'is' is not the 'is' of identity [See TC[1-L]a.] This solution could be paraphrased: 'g(13)' $\in$ nonsense because $13 \notin \mathcal{D}_g$ [If so paraphrased, 'nonsense' is used as a name for the class of meaningless expressions.] Also, note that equally good answers to parts (e) and (f) are:

(e) $\mathcal{D}_g = \{5, 7, 3, 8\}$ (f) $\mathcal{R}_g = \{9, 3, 7, 4\}$

*

Answers for Part B [on pages 5-57, 5-58, and 5-59].

1. (a) 18 (b) 1 (c) 27 (d) 'f(11)' is nonsense
   (e) $\{1, 2, 3, 4, 5\}$ (f) $\{1, 4, 11, 18, 27\}$

2. (a) 8 (b) $[1.2 \notin \mathcal{D}_F]$ (c) 7 (d) 9
   (e) $\{1.1, 1.3, 1.6, 2\}$ (f) $\{7, 8, 9, 10\}$

3. (a) 16 (b) −5 (c) 10.9 (d) $3\pi + 1$
   (e) the set of real numbers (f) the set of real numbers

4. (a) 3 (b) 17 (c) $[3.2 \notin \mathcal{D}_f]$ (d) 11 (e) $[14 \notin \mathcal{R}_f]$
   (f) $[3.5 \notin \mathcal{R}_f]$ (g) the set of integers (h) the set of odd numbers

5. (a) 1/6 (b) 6 (c) −1/2 (d) 2 (e) $[0 \notin \mathcal{D}_g]$
   (f) $[0 \notin \mathcal{R}_g]$ (g) $\{x: x \neq 0\}$ (h) $\{x: x \neq 0\}$

6. (a) 7 (b) 7 (c) 7 (d) 7
   (e) [Any numeral for a real number is a correct answer.]
   (f) $[0 \notin \mathcal{R}_H]$ (g) the set of real numbers (h) $\mathcal{R}_H = \{7\}$

The word 'mapping' suggests maps of geographical regions. The analogy between "using" a function to determine a correspondence of the members of one set with those of another, and the process of drawing a map of a country, may be helpful. But, like most analyses, it can be misleading if accepted uncritically. The map-drawing process consists of choosing patches of a sheet of paper to correspond with geographical objects and doing some art work on the paper to delineate the chosen patches. The final result is a conventionalized picture of the country, and this is called a map. In such a case, there is a function which determines the chosen correspondence of certain patches of paper with certain geographical objects. Its ordered pairs have geographical objects as first components, and patches of paper as second components. Its domain is a set of geographical objects, and its range is a set of patches of paper. Now, one might be tempted to think of the range of this function as a map. But a map differs from the range of this function in just the way that a graph of an open sentence differs from the solution set of the sentence [see TC[3-11, 12]]. So, when one speaks of a function as determining a mapping, one thinks of a correspondence which might be used in drawing a map, rather than of a map. [Notice that when one graphs functions whose domain and range are sets of real numbers, one is drawing maps of the number plane. Here the number plane plays the role of the country to be mapped, and the functions play the roles of some of the geographical objects. In this case the function which determines the mapping has functions as arguments and patches of graph-paper as values.]

[As stated in this text, a function is said to map its domain on its range. For your information, it is also customary to say that a function maps its domain in any set which includes the range of the function. This terminology being standard, do not allow students to slur 'on's to 'in's.]

*

The answer to the bracketed question is 'No'.

Since it is customary to use numerals, rather than names of ordered pairs, in labeling graphs of axes, students may not be awake to the fact that all points on a picture of the number plane, including those on the graphs of the axes, are graphs of ordered pairs. Be sure, in discussing Figure 1, that they are fully aware of this fact before passing to the text below the figure. You might, for example, point out that in using Figure 1 to find the value of g corresponding with a given argument a, one begins by locating the graph of the ordered pair whose first component is a and whose second component is 0. [We don't, <u>in using Figure 1,</u> begin by finding "the graph of a on the graph of the x-axis". But, in Figure 2, we <u>do</u> look for the graph of a on the graph of $\mathcal{D}_g$.]

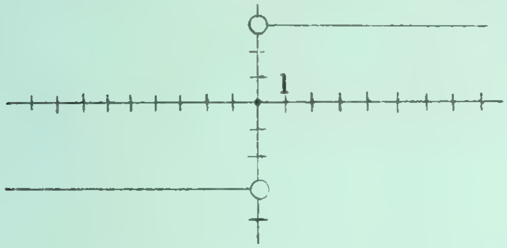
*

Compare Figure 3 on page 5-63 with the diagram on page 5-36.

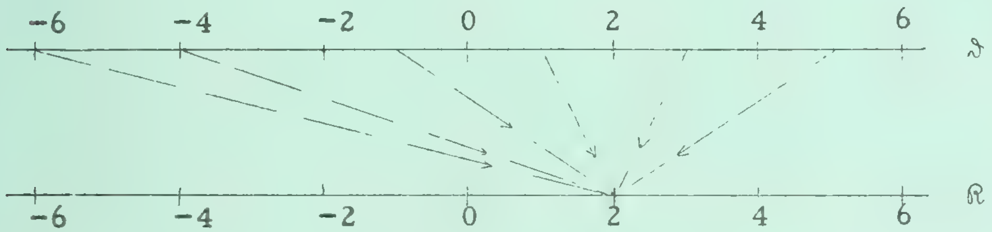
Correction.	Delete from line 9b:
	[What do we call such a mapping?]

Answers to questions in the text on page 5-64.

$f = \{(x, y): y = 2x\}$



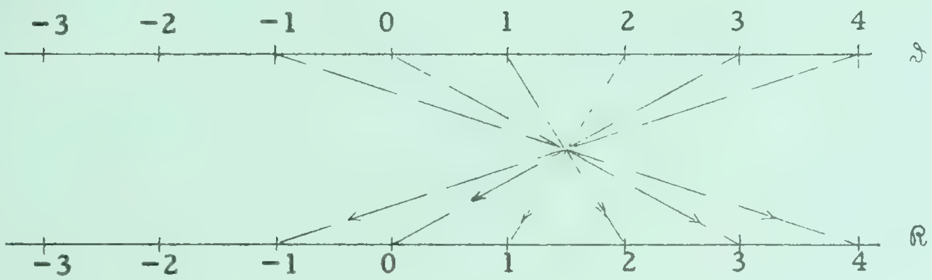
$\{(x, y): y = 2\}$



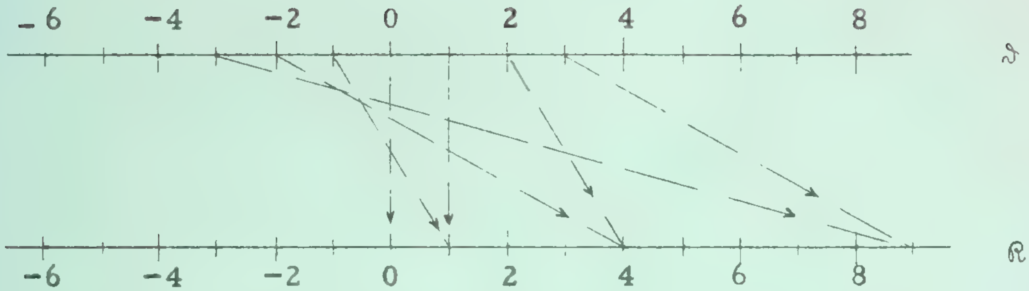
*

Answers for Part A.

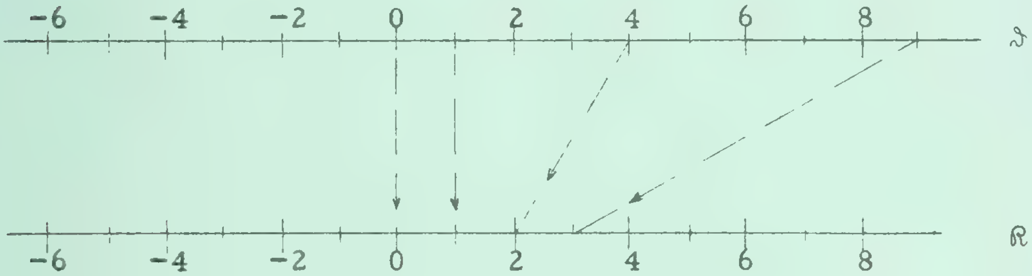
1.



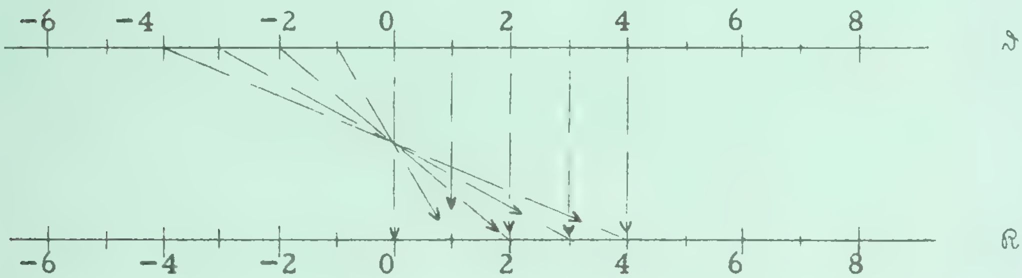
2.


-6 -4 -2 0 2 4 6 8
𝒟
-6 -4 -2 0 2 4 6 8
ℛ

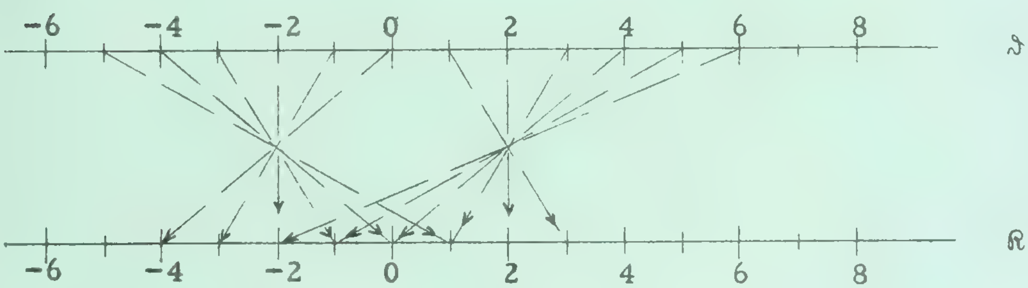
3.


-6 -4 -2 0 2 4 6 8
𝒟
-6 -4 -2 0 2 4 6 8
ℛ

4.


-6 -4 -2 0 2 4 6 8
𝒟
-6 -4 -2 0 2 4 6 8
ℛ

5.


-6 -4 -2 0 2 4 6 8
𝒟
-6 -4 -2 0 2 4 6 8
ℛ

Answers for Part B [on pages 5-65 and 5-66].

1. (a) 5 (b) 8 (c) 6

2. (a) 8 (b) 3 (c) 1, 8

3. (a) 12 (b) 4 (c) [$3 \notin \mathcal{D}_h$]

   [The fact that 3 is listed in the left loop along with 1, 2, 4, and 5 tells you only that the domain of the mapping is a subset of $\{1, 2, 3, 4, 5\}$. In the case of finite mappings which are pictured this way, we depend on the arrows to tell us which elements are members of the domain. In this connection, note the mapping f pictured at the top of page 5-70. In that mapping, the domain of f is $\{5, 6, 7, 8\}$.]

4. (a) -7 (b) π (c) π

5. (a) 9 (b) [$-2 \notin \mathcal{D}_g$] (c) ${}^{+}\sqrt{2}$

6. (a) 2 (b) [$7 \notin \mathcal{R}_g$] (c) 14
   (d) [$3.5 \notin \mathcal{D}_h$] (e) -10 (f) 0

7. (a) 2.5 (b) -3.5 (c) 0
   (d) 18.6 (e) -12.2 (f) 4/3

8. (a) 3 (b) 3 (c) -4
   (d) 0 (e) -1 (f) each number in $\{x: 8 \le x < 9\}$

Correction. On page 5-67, in line 3b, change '$h \geq 0$' to '$y \geq 0$'.

If one uses, for example, 'x^2' as a name for the squaring function, he introduces a convention which is not consistent with the use of 'x' as a variable. 'x^2' is already a pronumeral expression which has values but does not name anything, and to use it also as a name breeds confusion. For example, if one uses 'x^2' as a name for $\{(x, y): y = x^2\}$ then, since, for example, $\{(x, y): y = x^2\} = \{(y, x): x = y^2\}$, there is no apparent reason for not using 'y^2' as a name for the same function. And, if 'x^2' and 'y^2' are used as names for the same thing, one must admit '$x^2 = y^2$' as a true statement. But, in other connections, one needs to consider '$x^2 = y^2$' to be an open sentence, neither true nor false. A similar objection applies to the use, say, of '$f(x)$' as a name for a function f.

An objection to '$y = x^2$' as a name for the squaring function is that it involves using a sentence as a noun. And, as above, it would lead one to accept '$y = x^2 = x = y^2$' as a true statement in which the middle '=' is a verb while the other two '='s are "letters" occurring in two nouns. This is confusing in itself and also conflicts with a well-established and useful convention according to which '$y = x^2 = x = y^2$' is an abbreviation for the open sentence '$y = x^2$ and $x^2 = x$ and $x = y^2$'.

One might attempt to meet these objections by saying that the phrase 'the function x^2' is not the expression 'x^2' and, for example, the sentence 'the function x^2 = the function y^2' need lead to no confusion. But, in practice, the use of languages like (1) [on page 5-66] usually leads to statements such as 'x^2 is an example of a function' and 'Now, we shall study such functions as $y = x^2$, $y = 3x^2 - 5$, and $2 = x^2 + 2y - 3$'. In both of these, pronumeral expressions or sentences are evidently being used as nouns. The modes of expression illustrated in (4), (5), and (6) [on page 5-67] do not so readily foster illegitimate abbreviations as do (1), (2), and (3).

Answers for questions at bottom of page 5-67.

f_1 is not defined at 2

f_2 is not defined at 1 nor at -1

f_3 is defined neither at 1 nor at -2

f_4 is undefined at any negative number

f_5 is not defined at any negative number

f_6 is defined only for nonpositive real numbers

f_7 is defined only for nonnegative real numbers

Answers for Part A.

1. $\{(x, y): y = 5 - x\}$

2. $\{(x, y): y = 1 + 2x^2\}$

3. $\{(x, y), x > 0: y = 3x + 4\}$, or: $\{(x, y): x > 0 \text{ and } y = 3x + 4\}$

4. $\{(x, y), x \geq 0: y = 1 + \sqrt{x}\}$, or: $\{(x, y): y \geq 1 \text{ and } (y - 1)^2 = x\}$

*

When discussing Sample 2, do not allow students to speak as though f were two functions. A single function, f, is described by stating a rule for finding its value for each of its arguments. The rule has two parts, but f is one function. If students have difficulty with this, point out that, for example, in Exercise 15 on page 5-51 they had no doubt that the graph shown there pictures a single function, although a written rule for finding its values would require several parts. Also point out that the function of Exercise 1 in Part A on page 5-57 can be described, calling it 'g', by:

$$g(x) = \begin{cases} 4, & \text{for } x = 3 \text{ or } 17 \quad [\text{i.e., } x = 3 \text{ or } x = 17] \\ 8, & \text{for } x = 5 \text{ or } 16 \end{cases}$$

On the other hand, point out that the function f of Sample 2 is the union of two other functions. In fact, if $f_1 = \{(x, y): x \geq 0 \text{ and } y = 2\}$ and $f_2 = \{(x, y): x < 0 \text{ and } y = -2\}$ then $f = f_1 \cup f_2$. But the same applies to any function [other than $\emptyset$]. For example,

$$\{(x, y): y = x^2\} = \{(x, y): x \leq 3 \text{ and } y = x^2\} \cup \{(x, y): x > 3 \text{ and } y = x^2\}.$$

*

5. $\{(x, y): (x \geq 1 \text{ and } y = x) \text{ or } (x < 1 \text{ and } y = 2 - x)\}$

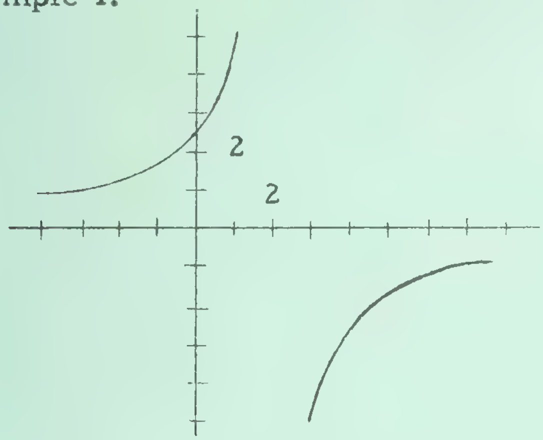
6. $\{(x, y): (x > 3 \text{ and } y = 3) \text{ or } (-3 \leq x \leq 3 \text{ and } y = x) \text{ or } (x < -3 \text{ and } y = -3)\}$

7. $\{(x, y): -2 \leq x \leq 2 \text{ and } y = x + 3\}$

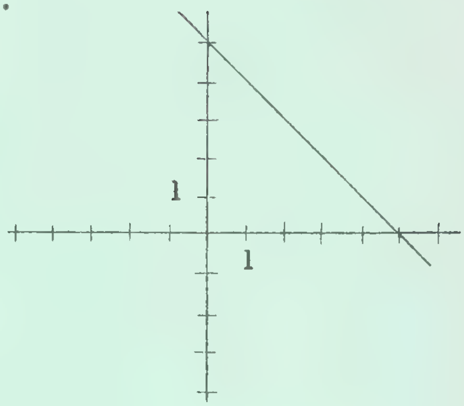
8. $\{(x, y): (x \text{ is rational and } y = -1) \text{ or } (x \text{ is irrational and } y = 1)\}$

Answers for Part B [on page 5-68].

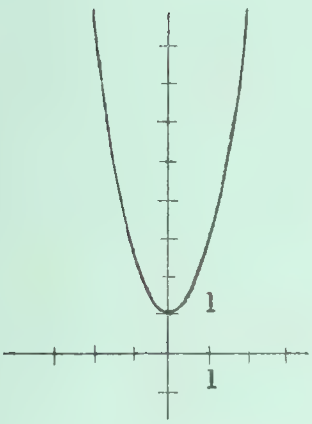
Sample 1.



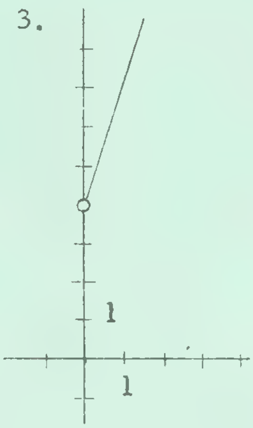
1.



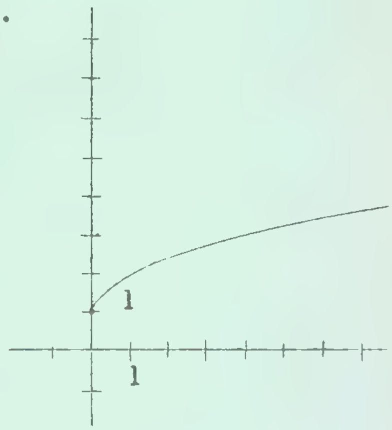
2.



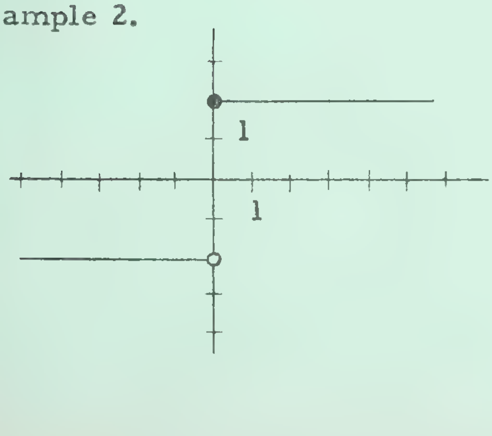
3.



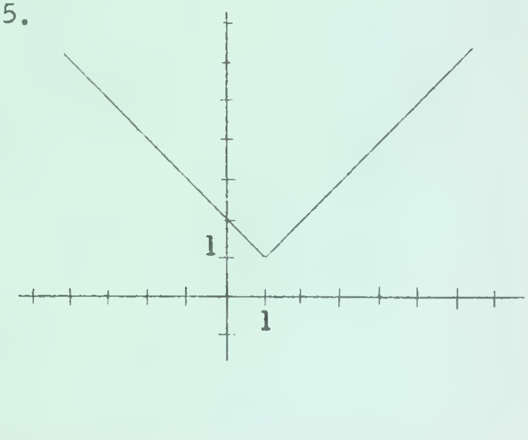
4.



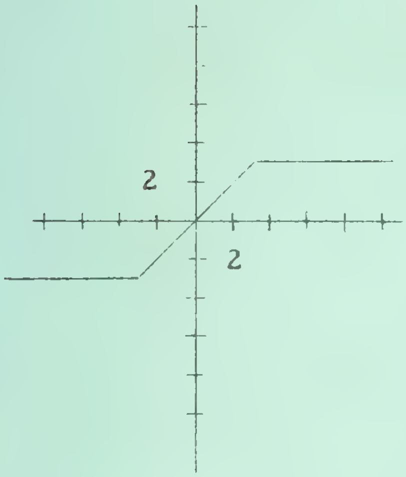
Sample 2.



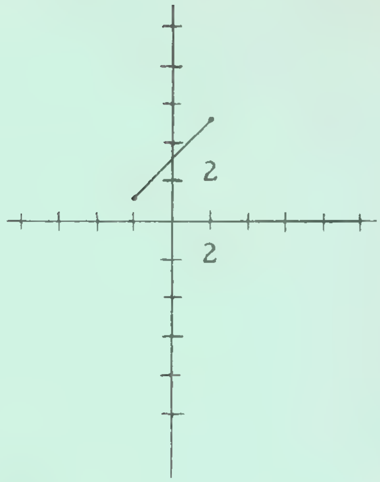
5.



6.



7.



[The function of Exercise 8 cannot be graphed. If you tried, you would get a pair of parallel "lines", with each "line" full of holes.]

Answers for Exploration Exercises [on pages 5-69, 5-70, and 5-71].

1. (a) 4 (b) 12 (c) 4 (d) 12 (e) [$F_2(4) \notin \mathfrak{D}_{F_1}$]
   (f) 3 (g) [$F_1(3) \notin \mathfrak{D}_{F_1}$] (h) 4

2. (a) -3 (b) -3 (c) 1 (d) 1 (e) -3
   (f) [$-3 \notin \mathfrak{D}_f$] (g) 5 (h) 5 (i) 6 (j) [$f(5) \notin \mathfrak{D}_f$]

3. (a) 12 (b) 87 (c) -7 (d) 5
   (e) 7 (f) 29 (g) -13 (h) 2.7

4. (a) 19 (b) 3 (c) 33 (d) 5 (e) 100
   (f) 1002 (g), (h), (i), (j) the set of real numbers

   [While solving Exercise 4, students should discover that, for each real number x, $H(G(x)) = x = G(H(x))$. Exercise 4 helps prepare for the work on inverses of functions which begins on page 5-79.]

5. (a) 8 (b) 6 (c) 7 (d) [$8 \notin \mathfrak{D}_g$] (e) 3
   (f) [$1 \notin \mathcal{R}_f$] (g) 5 (h) 3 (i) 3, 5 (j) [$7 \notin \mathfrak{D}_g$]
   (k) 9 (ℓ) [$1 \notin \mathcal{R}_f$]

6. (a) (1, 7), (2, 7), (3, 9), (4, 6), (5, 4)
   (b) (7, 5), (9, 5), (4, 8), (6, 6) (c) {4, 6, 7, 9}; {4, 6, 7, 9}
   (d) (1, 5), (2, 5), (3, 5), (4, 6), (5, 8)

7. (a) (1, 3), (2, 7), (3, 7), (4, 5), (5, 8)
   (b) (3, 2), (7, 7), (9, 4), (5, 4), (8, 1), (2, 3)
   (c) {3, 5, 7, 8}; {2, 3, 5, 7, 8, 9}
   (d) (1, 2), (2, 7), (3, 7), (4, 4), (5, 1)

8. (a) $(-1, 1)$, $(-2, 1)$, $(-3, 2)$, (3, 3), (2, 4), (1, 8)
   (b) (1, 10), (2, 11), (3, 11), (5, 12), (6, 13)
   (c) {1, 2, 3, 4, 8}; {1, 2, 3, 5, 6}
   (d) $(-1, 10)$, $(-2, 10)$, $(-3, 11)$, (3, 11)

Quiz.

In each of the following exercises you are given a function. Fill in the blanks to make true sentences.

1. $f = \{(1, 9), (-2, 5), (7, 3), (0, 1), (3, 7)\}$

   (a) $f(1) =$ ______ (b) $f(3) =$ ______ (c) $f(f(3)) =$ ______

   (d) $f($____$) = 5$ (e) $f($____$) = 1$ (f) $f(f($____$)) = 9$

2. $g = \{(x, y): \; y = 3x - 4\}$

   (a) $g(1) =$ ______ (b) $g(5) =$ ______ (c) $g($____$) = 8$

   (d) $g($____$) = -10$ (e) $g(0) =$ ______ (f) $g(g(0)) =$ ______

3. $h(x) = x^2 + 3, \quad \mathcal{D}_h = \{-3, -2, -1, 0, 1, 3\}$

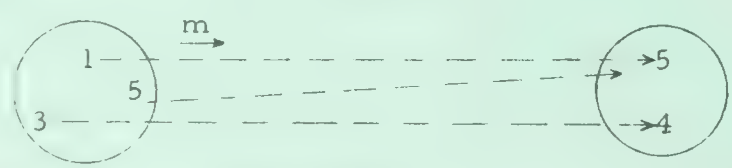
   (a) $h(3) =$ ______ (b) $h(-3) =$ ______ (c) $h($____$) = 3$

   (d) $h($____$) = 7$ (e) $h($____$) = -3$ (f) $h(h(0)) =$ ______

4. k is the function which maps each real number on its triple.

   (a) $k(8) =$ ______ (b) $k(7.2) =$ ______ (c) $k(-3.3) =$ ______

   (d) $k($____$) = 81$ (e) $k(k($____$)) = 81$ (f) $k(k(k($____$))) = 81$

5. 

   (a) $m(1) =$ ______ (b) $m(5) =$ ______ (c) $m($____$) = 4$

   (d) $m(m(1)) =$ ______ (e) $m($____$) = 5$ (f) $m($____$) = 3$

*

Answers for Quiz.

1.	(a) 9	(b) 7	(c) 3	(d) -2	(e) 0	(f) 0
2.	(a) -1	(b) 11	(c) 4	(d) -2	(e) -4	(f) -16
3.	(a) 12	(b) 12	(c) 0	(d) -2	(e) $[-3 \notin \mathcal{R}_h]$	(f) 12
4.	(a) 24	(b) 21.6	(c) -9.9	(d) 27	(e) 9	(f) 3
5.	(a) 5	(b) 5	(c) 3	(d) 5	(e) 1 and 5	(f) $[3 \notin \mathcal{R}_m]$

The Exploration Exercises have suggested that, to each ordered pair (f, g) of functions there corresponds a function h defined by:

$$h(x) = f(g(x)), \text{ for each } x \in \mathcal{D}_g \text{ such that } g(x) \in \mathcal{D}_f$$

The function h so defined is called the composition [sometimes: the compositum] of f with g, and is usually denoted by '$f \circ g$'. Note that $\mathcal{D}_{f\circ g} = \{x \in \mathcal{D}_g : g(x) \in \mathcal{D}_f\}$ and, so, is always a subset of $\mathcal{D}_g$. Exercise 8 on page 5-71 shows that $\mathcal{D}_{f\circ g}$ may be a proper subset of $\mathcal{D}_g$. This occurs precisely when there are values of g which are not arguments of f. That is, $\mathcal{D}_{f\circ g} = \mathcal{D}_g$ if and only if $\mathcal{R}_g \subseteq \mathcal{D}_f$. It may even happen [it is easy to make up examples] that no value of g is an argument of f. In this case $\mathcal{D}_{f\circ g} = \emptyset$, and, also, $f\circ g = \emptyset$.

Composition of functions is a binary operation on functions, just as multiplication of numbers is a binary operation on numbers. As is shown by the example on page 5-73, composition is not commutative. However, as will be seen shortly [page 5-78], composition is associative. Other similarities and differences with multiplication of numbers will come up later in the unit.

The operation of composing functions is a special case of an operation on relations in general. For example, x is a grandson of z if and only if there is a person y such that x is a son of y and y is a child of z. So, the relation of being-a-grandson-of is the result of "composing" the relation of being-a-son-of with the relation of being-a-child-of. This more general kind of composition is called 'relative multiplication'. As another example, the relation of being-a-son-in-law-of is the result of composing the relation of being-the-husband-of with that of being-a-daughter-of.

*

Up to now, names of functions have been, for the most part, uncomplicated symbols such as 'F', 'g', and 'h_2'. Now that more complex symbols, like '$f\circ g$', are being used as function names it sometimes makes reading simpler if one encloses such function names in brackets. This has been done in the last line of the second paragraph on page 5-73 where we have the sentence '$[q\circ p](3) = 5$'. Brackets are also used in the definition at the top of page 5-74, in the Solution to the Sample on page 5-75, and in many of the exercises in later pages.

One way of explaining the brackets in such expressions as '$[q\circ p](3)$' on page 5-73 is to point out that there might be someone who regarded,

say, the symbol '$q \circ p(3)$' as ambiguous.
The symbol:

$$[\_\_\_](\_\_\_)$$

is a more adequate symbol to use in indicating application of a function to an argument than is the customary symbol:

$$\_\_\_\ (\_\_\_)$$

We shall continue to use the more familiar abbreviated notation '$\_\_\_\ (\_\_\_)$' in contexts where the function name is not a combination of symbols.

Answer for bracketed question: $\mathcal{D}_{f \circ g} = \mathcal{D}_g$ if and only if $\mathcal{R}_g \subseteq \mathcal{D}_f$
[For, $\{x \in \mathcal{D}_g : g(x) \in \mathcal{D}_f\} = \mathcal{D}_g$ if and only if, for each $x \in \mathcal{D}_g$, $g(x) \in \mathcal{D}_f$.]

*

Answers for Part A [on pages 5-74, 5-75, and 5-76].

1. (6, 4), (9, 16), (12, 1), (5, 9)   2. (John, 22), (Emma, 25)

3. (a) (2, 3), (3, 6), (5, 3), (6, 2), (8, 6)

   (b) (0, 1), (1, 2), (2, 0), (3, 2)

4. (a) (0, 5), (1, 3), (3, 9), (4, 11)

   (b) (0, 2), (2, 8)

5. $t \circ s$ is the perimeter of an equilateral triangle. That is, $t \circ s$ is the function which maps each equilateral triangle on its perimeter. Your students may give a brace-notation name, such as:

   $\{(x, y) \in T \times N$: y is the triple of the side-measure of $x\}$, where T is the set of equilateral triangles

   or:

   $\{(x, y) \in T \times N$: y is the perimeter of $x\}$, where T is the set of equilateral triangles

   Both of these are correct. However, the latter is preferable, because it indicates that the student understands the meaning of the word 'perimeter' as it applies to equilateral triangles.

> **Correction.** In the first line of the solution to the sample, insert a comma after '$\mathcal{D}_g$'.

6. (a) $\{(x, y)\colon\ y = 81x^4\}$
   (b) $\{(x, y)\colon\ y = 3x^4\}$

7. (a) $\{(x, y)\colon\ y = |x - 5|\}$
   (b) $\{(x, y)\colon\ y = |x| - 5\}$

8. (a) $(x + 3)^2$
   (b) $x^2 + 3$

9. (a) x
   (b) x

10. (a) 0 (b) 16 (c) 1 (d) 1 (e) 5, -5 (f) $-4 \notin \mathcal{R}_{q \circ p}$

11. (a) 36 (b) 8 (c) 0 (d) 32 (e) 2, -4 (f) 2

12. (a) $2t^2 + 1$
    (b) $(2t - 1)^2 + 1$

13. (a) $2t^2 + 1$
    (b) $\begin{cases} (2t - 1)^2 + 1, & \text{for } t \geq 1 \\ (7 - t)^2 + 1, & \text{for } t < 1 \end{cases}$

14. (a) 7
    (b) 11

☆15. (a) $\{(x, y),\ x > 0\colon\ y = -\sqrt{2x}\}$
    (b) $\emptyset$

16. (a) $\{(4, 4), (5, 5), (6, 6)\}$ [or: $\{(x, y),\ x \in \mathcal{D}_g\colon\ y = x\}$]
    (b) $\{(3, 3), (7, 7), (-1, -1)\}$ [or: $\{(x, y),\ x \in \mathcal{D}_f\colon\ y = x\}$]

17. (a) $\{(x, y)\colon\ y = x\}$
    (b) $\{(x, y)\colon\ y = x\}$

18. (a) $\{(x, y)\colon\ y = 9x + 28\}$
    (b) $\{(x, y)\colon\ y = 27x + 91\}$

19. (a) $\{(x, y)\colon\ y = 4x - 3\}$
    (b) $\{(x, y)\colon\ y = -8x + 9\}$

20. the perimeter of x [or: $\{(x, y) \in S \times N\colon$ y is the perimeter of $x\}$]; the set of all squares

☆21. (a) the father-in-law of x; the set of married people
    (b) the mother of x; the set of married people

(c) To show that the sentence 'x belongs to the domain of $f \circ [g \circ h]$' is equivalent to (*):

By definition,

$$\begin{gathered} x \text{ belongs to the domain of } f \circ [g \circ h] \\ \text{if and only if} \\ x \in \mathfrak{D}_{g \circ h} \text{ and } [g \circ h](x) \in \mathfrak{D}_f. \end{gathered}$$

But, again by definition,

$$\begin{gathered} x \in \mathfrak{D}_{g \circ h} \\ \text{if and only if} \\ x \in \mathfrak{D}_h \text{ and } h(x) \in \mathfrak{D}_g. \end{gathered}$$

Also, by definition, for each $x \in \mathfrak{D}_h$ such that $h(x) \in \mathfrak{D}_g$,

$$[g \circ h](x) = g(h(x)).$$

So,

$$\begin{gathered} x \text{ belongs to the domain of } f \circ [g \circ h] \\ \text{if and only if} \\ [x \in \mathfrak{D}_h \text{ and } h(x) \in \mathfrak{D}_g] \text{ and } g(h(x)) \in \mathfrak{D}_f. \end{gathered}$$

Therefore, since conjunction is associative, it follows that the sentence 'x belongs to the domain of $f \circ [g \circ h]$' is equivalent to (*).

> Correction. In the last line of part (b) on page 5-78, change
>
> '$[f\circ]g\circ h]](x)$' to '$[f\circ[g\circ h]](x)$'.
>
> ↑

Answers for Part B.

In each of the four blanks: $\{(1, 2), (2, 0), (3, 4)\}$

$f\circ g = f_1\circ g$ if and only if $f_3 \subseteq f$

There are infinitely many such functions.

The only such function whose domain is $\mathcal{R}_g$ is f_3.

*

Answers for Part C [on pages 5-77 and 5-78].

1. The example on page 5-73 has already shown that composition is not commutative. Also, the Sample on page 5-75 and each of the exercises of Part A [on pages 5-74 ff.], with the exception of Exercises 9 and 17, show that the result of composing two functions may depend on which function is composed with which.
   There are some interesting special cases:

   If $f = \{(x, y): y = ax + b\}$ and $g = \{(x, y): y = cx + d\}$, then $f\circ g = g\circ f$ if and only if $(a - 1)d = (c - 1)b$.

   If $f = \{(x, y): y = x^2\}$ and $g = \{(x, y): y = x^3\}$, then $f\circ g = g\circ f$. [Similarly for other "power functions".]

2. (a) 4 (b) 18 (c) 18 (d) 18 (e) 13 (f) 13
   (g) 26 (h) 26 (i) 22 (j) 22 (k) $2(x + 5)$
   (ℓ) $2[(x - 3) + 5]$ (m) $(x - 3) + 5$ (n) $2[(x - 3) + 5]$

3. (a) $f(g(h(x)))$; $f(g(h(x)))$ (b) $g(h(x))$; $f(g(h(x)))$

Quiz.

Suppose that

$f = \{(0, 3), (4, 7), (5, 8)\}$, $g = \{(5, 0), (9, 4), (10, 5)\}$,
$h = \{(5, 0), (9, 4), (10, 5), (12, 7)\}$, $j = \{(x, y): y = x - 5\}$,
$k = \{(5, 0), (9, 4), (10, 5), (12, 5)\}$, and $\ell = \{(x, y): y = x + 3\}$.

Fill in the blanks.

1. $[f \circ g](5) =$ ______ 2. $[f \circ g](9) =$ ______ 3. $[f \circ g](10) =$ ______
4. $f \circ g = \{$______________________$\}$
5. $[f \circ h](5) =$ ______ 6. $[f \circ h](9) =$ ______ 7. $[f \circ h](10) =$ ______
8. $f \circ h = \{$______________________$\}$
9. $f \circ j = \{$______________________$\}$
10. $[f \circ k](5) =$ ______ 11. $[f \circ k](9) =$ ______ 12. $[f \circ k](10) =$ ______
13. $[f \circ k](12) =$ ______ 14. $f \circ k = \{$______________________$\}$
15. $[\ell \circ j](5) =$ ______ 16. $[\ell \circ j](9) =$ ______ 17. $[\ell \circ j](10) =$ ______
18. $[\ell \circ j](15) =$ ______ 19. $j(a) =$ ______ 20. $\ell(b) =$ ______
21. $\ell(a - 5) =$ ______ 22. $j(r) =$ ______ 23. $[\ell \circ j](r) =$ ______
24. $\ell \circ j = \{$______________________$\}$ 25. $g(5) =$ ______
26. $[j \circ g](5) =$ ______ 27. $[\ell \circ [j \circ g]](5) =$ ____ 28. $[[\ell \circ j] \circ g](5) =$ ____

*

Answers for Quiz.

1. 3 2. 7 3. 8 4. (5, 3), (9, 7), (10, 8) 5. 3
6. 7 7. 8 8. (5, 3), (9, 7), (10, 8) 9. (5, 3), (9, 7), (10, 8)
10. 3 11. 7 12. 8 13. 8
14. (5, 3), (9, 7), (10, 8), (12, 8) 15. 3 16. 7 17. 8
18. 13 19. $a - 5$ 20. $b + 3$ 21. $a - 2$ 22. $r - 5$ 23. $r - 2$
24. $(x, y): y = x - 2$ 25. 0 26. -5 27. -2 28. -2

In discussing the questions on page 5-79, it may be helpful to draw graphs of the functions referred to.

(1) $(0, -5)$, $(3/2, -7/2)$, $(-3, -8)$, . . .

(2) $(-5, 0)$, $(-7/2, 3/2)$, $(-8, -3)$, . . .

(3) yes	(4) yes	(5) yes	(6) no
(7) no	(8) yes	(9) no	(10) yes

A function has at most one inverse.

If a function has an inverse then the converse of this inverse is the given function. So, if a function has an inverse then the function is, itself, the inverse of its inverse.

If g is the inverse of f then $\mathcal{D}_g = \mathcal{R}_f$ and $\mathcal{R}_g = \mathcal{D}_f$.

$\{(x, y): y = x\}$ is its own inverse. So is $\{(x, y): xy = 1\}$.
[In general, a function is its own inverse just if it is a function which is a symmetric relation.]

If g is the inverse of f then $g \cup f$ is a symmetric relation whose domain and range are the set $\mathcal{D}_f \cup \mathcal{R}_f$.

Here is an an alternative solution for Sample 2.

Solution. A function has an inverse if no two of its ordered pairs have the same second component. We shall show that f has an inverse by showing that no two of its ordered pairs have the same second component. That is, we shall show that if $(a, c) \in f$ and $(b, c) \in f$ then $a = b$.

Suppose that $(a, c) \in f$ and $(b, c) \in f$. Then $c = 3a - 5$ and $c = 3b - 5$. So, $3a - 5 = 3b - 5$. Hence, $3a = 3b$, and [since $3 \neq 0$] $a = b$.

Consequently, f has an inverse.

Unlike the solution in the text, this alternative solution does not furnish a description of the inverse of f. However, it illustrates the fundamental procedure generally used to determine whether or not a function does have an inverse. This procedure is used, for example, on TC[3-108, 109]a to show [proof of generalization (ii)] that the function $\{(x, y): x > 0 \text{ and } y = x^2\}$ has an inverse and, so, to justify introducing the function name '$\sqrt{\ }$'. [In further explanation, note that while, by definition, $\{(x, y): y > 0 \text{ and } x = y^2\}$ is the converse of $\{(x, y): x > 0 \text{ and } y = x^2\}$, we were not justified in introducing a function name [$\sqrt{\ }$] for this relation until we had proved generalization (ii) and, so, knew that the relation in question is a function.]

*

Answers for Part A [which begins on page 5-79].

1. Since the converse of f is $\{(6, -5), (7, -6), (-10, -7), (1/3, 1/2)\}$ and since this relation is a function, the function f has an inverse. Or: since no two ordered pairs in the function f have the same second component, f has an inverse.

2. F does not have an inverse since it contains two ordered pairs, (2, 9) and (4, 9), which have the same second component.

3. The converse of g is $\{(x, y): x = 5\}$. This relation is not a function since it contains two ordered pairs, for example, $(5, -4)$ and $(5, \sqrt{2})$, which have the same first component. Hence, g does not have an inverse.

that f is a subset of the converse of g]. Similarly, to say that

(ii) for each $k \in \mathcal{D}_g$, $[f \circ g](k) = k$

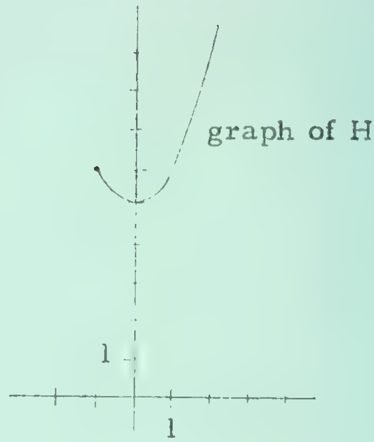
is to say that g is a subset of the converse of f.

So, (i) and (ii) together say that g is the converse of f. Since g is a function, this amounts to saying that f has an inverse, and that its inverse is g.

Notice that, since each subset of a function is a function, (i) alone tells us that the converse of f is a function--that is, that f has an inverse. But, to conclude that the inverse of f is g, we need additional information, for example, that $\mathcal{D}_g = \mathcal{R}_f$. Condition (ii), on the other hand, does not ensure that f has an inverse [although, as we have just seen, it does ensure that g does]. For example, (ii) is satisfied by taking for f the squaring function $\{(x, y): y = x^2\}$, and for g the square-rooting function $\{(x, y), x \geq 0: y = \sqrt{x}\}$. $[\forall_{x > 0} (\sqrt{x})^2 = x]$. However, since (ii) says that g is a subset of the converse of f, it follows from (ii) together with '$\mathcal{R}_g = \mathcal{D}_f$' that f has g as inverse. [Returning to (i), we see that if $f = \{(x, y), x \geq 0: y = \sqrt{x}\}$ and $g = \{(x, y): y = x^2\}$ then (i) is satisfied. Now, by definition of '$\sqrt{\ }$', $f = \{(x, y), x \geq 0: y \geq 0 \text{ and } y^2 = x\}$. Hence, the converse of f is $\{(x, y): x \geq 0 \text{ and } x^2 = y\}$, and is a proper subset of g. As predicted, the converse of f is a function. So, f has an inverse, and the inverse of f is a subset of g.]

10. [It may help to draw a graph of H.] H does not have an inverse because its converse contains two ordered pairs [for example, (6, −1) and (6, 1)] which have the same first component. The function h such that

$$h(x) = x^2 + 5 \text{ and } \mathcal{D}_h = \{x\colon x \geq 0\}$$



is a subset of H which does have an inverse. The inverse of h is the function g such that

$$g(x) = \sqrt{x - 5} \text{ and } \mathcal{D}_g = \{x\colon x \geq 5\}.$$

[Using brace-notation, $g = \{(x, y),\ x \geq 5\colon\ y = \sqrt{x - 5}\}$.]

Another subset of H which has an inverse is the function k such that

$$k(x) = x^2 + 5 \text{ and } \mathcal{D}_k = \{x\colon -1 \leq x \leq 0\}.$$

[Its inverse is $\{(x, y),\ x \geq 5\colon\ x \leq 6 \text{ and } y = -\sqrt{x - 5}\}$.] Each subset of k or of h is also a subset of H which has an inverse. Moreover, the union of a subset of k and a subset of h whose ranges have no member in common is a subset of H which has an inverse [and these are all]. So, one can obtain any subset of H which has an inverse by the following procedure: Choose a subset D_1 of $\{x\colon -1 \leq x \leq 0\}$ and a subset D_2 of $\{x\colon x > 0\}$ such that, for no member of D_1 is its opposite in D_2. Then a subset of H of the desired kind is

$$\{(x, y),\ x \in D_1\colon\ y = -\sqrt{x - 5}\} \cup \{(x, y),\ x \in D_2\colon\ y = \sqrt{x - 5}\}.$$

11. (a) first (b) second

12. (a) $\mathcal{D}_g = \mathcal{R}_f$ and $\mathcal{R}_g = \mathcal{D}_f$ (b) k [in both blanks]

*

Additional remarks about Exercise 12.--If f and g are functions [not necessarily inverse to one another] then, to say that

(i) for each $k \in \mathcal{D}_f$, $[g \circ f](k) = k$

is equivalent to saying that

for each $k \in \mathcal{D}_f$, if $(k, \ell) \in f$ then $(\ell, k) \in g$ [for each $\ell \in \mathcal{R}_f$].

But, this merely says that the converse of f is a subset of g [that is,

Correction. In line 17: $[g \circ f](k) = \underline{\quad\quad},$ $\uparrow$

4. The function h has an inverse. In fact, h is its own inverse.

5. $f = \{(x, y): y = 4x + 7\}$, and the converse of f is $\{(x, y): x = 4y + 7\}$. Since the latter is the <u>function</u> $\{(x, y): y = (x - 7)/4\}$, f has an inverse. In fact, the inverse of f is the function g such that $g(x) = (x - 7)/4$ and $\mathcal{D}_g$ is the set of real numbers.

*

When a function is described as in Exercise 5 and has an inverse, students should be required to give the same kind of description of the inverse of the given function. For example, to say that the function f of Exercise 5 has an inverse and its inverse is $\{(x, y): x = 4y + 7\}$, while correct, is not acceptable.

*

6. Similar to Exercise 5. The inverse of h is the function g whose domain is the set of real numbers and such that $g(x) = (3 - x)/2$.

7. $G = \{(x, y): x \geq 0 \text{ and } y = (5x + 1)/9\}$. The converse of G is $\{(x, y): y \geq 0 \text{ and } x = (5y + 1)/9\}$. But, this is the <u>function</u> $\{(x, y): y \geq 0 \text{ and } y = (9x - 1)/5\}$. So, G has an inverse. The inverse of G is the function H described by:

$$H(x) = (9x - 1)/5, \qquad \mathcal{D}_H = \{x: x \geq 1/9\}$$

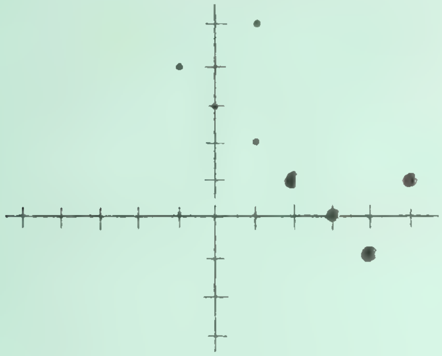
$[\forall_x\ [(9x - 1)/5 \geq 0 \iff x \geq 1/9]]$

8. F does not have an inverse since $(2, -1)$ and $(-2, -1)$ both belong to F. [That is, $(-1, 2)$ and $(-1, -2)$ both belong to the converse of F.]

9. G has an inverse. It is the function H such that $H(x) = (10 - x)/2$ [and $\mathcal{D}_H$ = the set of real numbers].

Answers for Part B [which begins on page 5-81].

1.

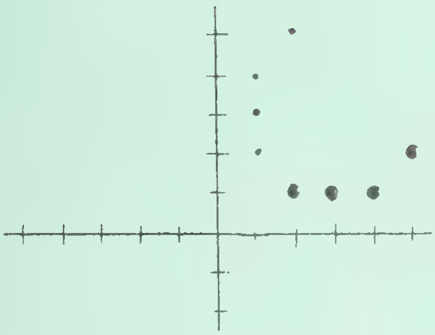


[Yes]

2.

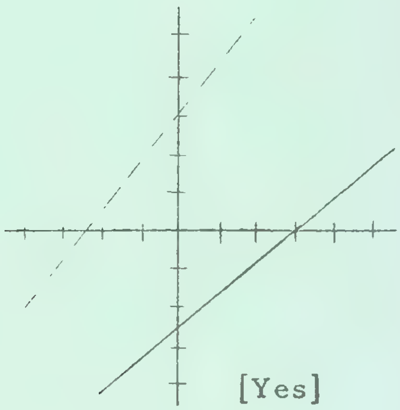
[Yes]

3.



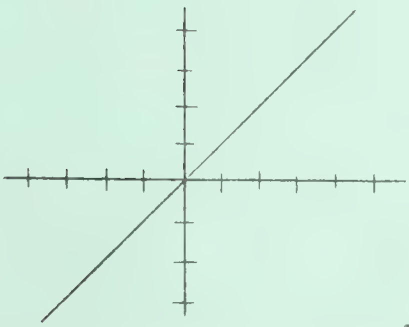
[No inverse]

4.



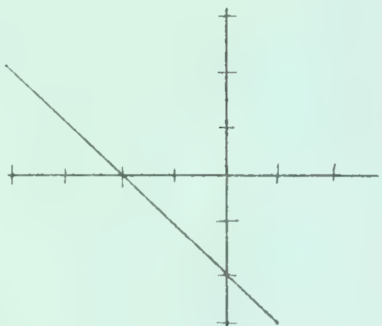
[Yes]

5.



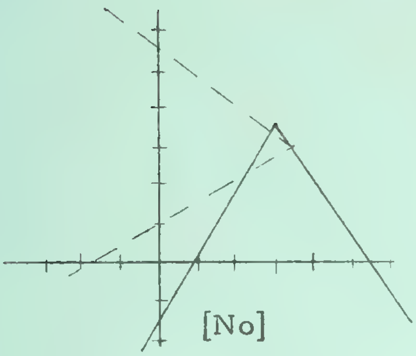
[Yes, it is its own inverse.]

6.



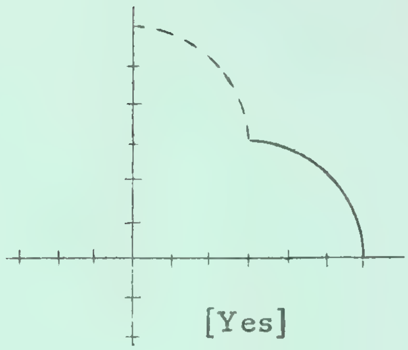
[Yes, it is its own inverse.]

7.



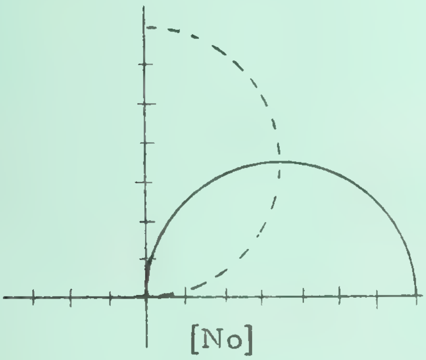
[No]

8.



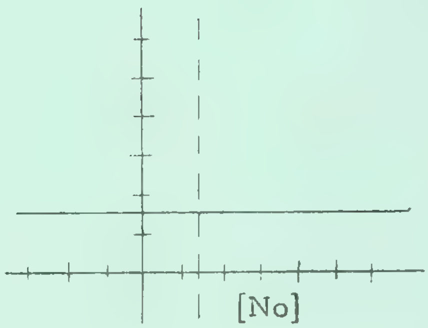
[Yes]

9.



[No]

10.



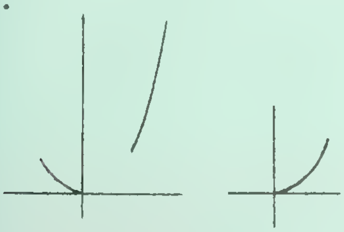
[No]
[A constant function does not have an inverse.]

*

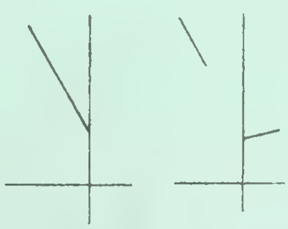
Answers for Part C [on pages 5-83 and 5-84].

For each exercise we show only two of several possible answers. [See TC discussion of Exercise 10 of Part A on page 5-81.] In each case we show one of the subsets whose range is the range of the given function and, except in Exercise 4, one of the subsets whose range is a proper subset of the range of the given function.

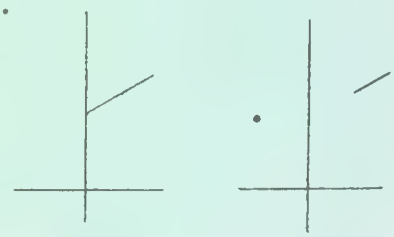
1.



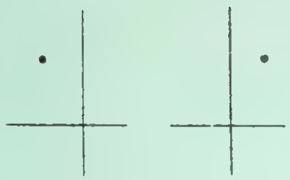
2.



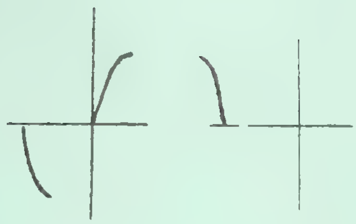
3.



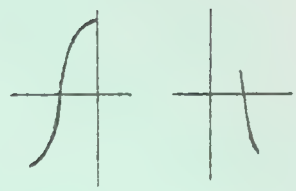
4.



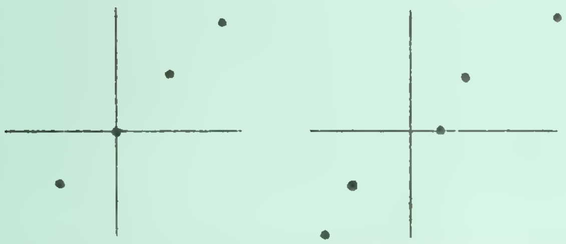
5.



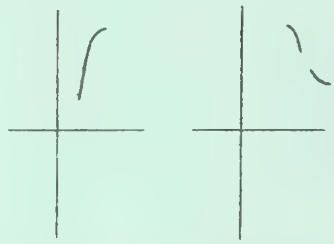
6.



7.



8.



9. Each relation R has a subset which is a function whose domain is the domain of the relation.

<u>Example</u> <u>1</u>. $R = \{(x, y):\ y > x\}$ $\qquad f = \{(x, y):\ y = x + 2\}$

$\mathfrak{D}_f = \mathfrak{D}_R$

<u>Example</u> <u>2</u>. $R = \{(3, 2), (5, 7), (4, 9), (3, 6), (8, 10)\}$

$f = \{(3, 2), (5, 7), (4, 9), (8, 10)\}$ or:

$\{(5, 7), (4, 9), (3, 6), (8, 10)\}$

$\mathfrak{D}_f = \{3, 5, 4, 8\} = \mathfrak{D}_R$

<u>Example</u> <u>3</u>. $R = \{(3, 2), (5, 7), (4, 9)\}$

$f = \{(3, 2), (5, 7), (4, 9)\}$

For any relation R, for each $x \in \mathfrak{D}_R$, there is at least one ordered pair in R with x as first component. Pick <u>one</u> of these ordered pairs. The set of such ordered pairs [one for each $x \in \mathfrak{D}_R$] is a function which is a subset of R and whose domain is $\mathfrak{D}_R$.

It follows that, for each function f, the converse of f has a subset which is a function whose domain is the domain of the converse of f. That is, whose domain is the range of f. The converse of such a subset is a subset of f which has an inverse whose domain is the range of f.

*

Students may need practice in using the inverse-notation in simple cases before they attack the problems in Part D. Use the functions f, h, and G described in Exercises 5, 6, and 7 of Part A on page 5-81.

1. $f^{-1}(2) =$ ______	2. $G^{-1}(1) =$ ______	3. $h^{-1}(-3) =$ ______
4. $f(G^{-1}(5)) =$ ______	5. $G^{-1}(f(2)) =$ ______	6. $h^{-1}(h(6)) =$ ______
7. $f(f^{-1}(-2)) =$ ______	8. $G^{-1}(G(\frac{1}{10})) =$ ______	9. $G(G^{-1}(\frac{1}{10})) =$ ___
10. $h(G^{-1}(f(\frac{1}{6}))) =$ ____	11. $f^{-1}(f^{-1}(f(2))) =$ ____	12. $G^{-1}(\pi) =$ ______

<u>Answers.</u>

1. $-\frac{5}{4}$	2. $\frac{8}{5}$	3. 3	4. $\frac{211}{5}$
5. $\frac{134}{5}$	6. 6	7. -2	8. $\frac{1}{10}$
9. $\frac{1}{10} \notin \mathcal{D}_{G^{-1}}$	10. $-\frac{121}{5}$	11. $-\frac{5}{4}$	12. $\frac{9\pi - 1}{5}$

*

The origin of the '$^{-1}$' notation for inverses is the analogy between function composition and multiplication of numbers. Just as, for each number $x \neq 0$, $x^{-1} \cdot x = 1$ and $x \cdot x^{-1} = 1$, so, for each function f which has an inverse, $f^{-1} \circ f = \{(x, y), x \in \mathcal{D}_f : y = x\}$ and $f \circ f^{-1} = \{(x, y) : x \in \mathcal{R}_f : y = x\}$. [See Exercise 12 of Part A, page 5-81.] The analogy just referred to is developed further on TC[5-115].

*

Alternate solution for Sample 1 of Part D.

$$F_2(6) = \frac{6}{3} - 1 = 1.$$

Since, for each x, $F_3^{-1}(x) = -\frac{x}{4}$,

it follows that $F_3^{-1}(F_2(6)) = F_3^{-1}(1) = -\frac{1}{4}$.

Quiz.

A. $R = \{(5, 9), (2, 6), (14, 18), (-2, 2), (6, -2)\}$

1. The converse of R is {____________________}.

2. Is R a function?

3. Is the converse of R a function?

4. Does R have an inverse?
   If it does, what is it? {______________________________}

5. Circle the meaningful expressions in the following list.

   $R(5)$, $R(18)$, $R^{-1}(9)$, $R^{-1}(14)$, $[R \circ R^{-1}](-2)$, $[R \circ R^{-1}](5)$

6. Give a simpler equivalent expression for each expression you circled in Exercise 5.

B. $f = \{(x, y): \; y = x + 3\}$

1. $f(2) =$ ______ 2. $f(5) =$ ______ 3. $f(\_\_) = 7$

4. $f^{-1}(7) =$ ______ 5. $f(b) =$ ______ 6. $f(\_\_) = a$

7. $f^{-1}(a) =$ ______ 8. $f^{-1} = \{(a, b): \; b =$ ________ $\}$

9. $f^{-1} = \{(x, y): \; y =$ _______ $\}$

*

Answers for Quiz.

A. 1. $(9, 5), (6, 2), (18, 14), (2, -2), (-2, 6)$ 2. Yes

3. Yes 4. Yes; $(9, 5), (6, 2), (18, 14), (2, -2), (-2, 6)$

5. $R(5)$, $R^{-1}(9)$, $[R \circ R^{-1}](-2)$ 6. $9, 5, -2$

B. 1. 5 2. 8 3. 4 4. 4 5. $b + 3$ 6. $a - 3$
7. $a - 3$ 8. $a - 3$ 9. $x - 3$

<u>Correction.</u> Exercise 15 should read: $5 \cdot F_{5}(5)$ ↑

Some students may prefer to find the inverse of each function before doing these exercises.

$F_1^{-1}(x) = \frac{x - 3}{2}$, $F_2^{-1}(x) = 3x + 3$, $F_3^{-1}(x) = -\frac{x}{4}$,

$F_4^{-1}(x) = x - 2$, $F_5^{-1}(x) = -\frac{x + 4}{2}$

*

Answers for Part D [which begins on page 5-84].

1. $\frac{1}{4}$	2. $-\frac{7}{3}$	3. $-\frac{8}{3}$	4. $\frac{5}{3}$	5. $\frac{28}{3}$
6. -1	7. $4r + 5$	8. $6t - 6$	9. $-\frac{9}{2}$	10. -3
11. -2	12. -2	13. $-\frac{4}{3}$	14. 5	15. -70
16. $-\frac{21}{2}$	17. $3a + 3$	18. $-\frac{5b + 1}{2}$	19. -1	20. 39
21. 5	22. $7a + 2a - 8$		23. 5	24. 5
25. 843	26. 927			

*

Correction. The third line of Sample 2 should read:

$$h = \{(3, 9), (5, 12), (8, 7), (7, 10)\}$$

Answers for Exploration Exercises [on pages 5-86, 5-87, 5-88, and 5-89].

1. $\{(1, 1), (2, 4), (3, 9), (-1, 1)\}$, or any function which contains the members of this set; for example, $\{(x, y): y = x^2\}$, and $\{(x, y), x \in I: y = x^2\}$.

2. Such a function would have to contain (8, 14) and (8, 18). So, there is no such function.

3. $\{(5, 7), (9, 12)\}$, or any function which contains the members of this set <u>and</u> whose domain does not contain 1 or 8. [If, for example, $1 \in \mathcal{D}_f$ then $(0, f(1)) \in f \circ g$. In this case, since $0 \notin \mathcal{D}_h$, $h \neq f \circ g$.]

4. Since $\mathcal{D}_h \not\subseteq \mathcal{D}_g$ [$4 \in \mathcal{D}_h$ and $4 \notin \mathcal{D}_g$], there is no such function.

5. There is no function f such that $f \circ g = h$. For, since $g(3) = 6$ and $h(3) = 9$, (6, 9) must belong to any such function f. So, $[f \circ g](8) = f(g(8)) = f(6) = 9$. Hence $8 \in \mathcal{D}_{f \circ g}$. But, $8 \notin \mathcal{D}_h$.

*

In Samples 1 and 2 and Exercises 1 and 2, $\mathcal{D}_h = \mathcal{D}_g$. <u>When this is the case</u> there is a function f such that $h = f \circ g$ if and only if, whenever g has the same value for two arguments, then h has the same value for these arguments. For, if this condition is satisfied then with each value of g there corresponds just one value of h, and the function f_1 whose members are obtained by pairing such corresponding values is such that $h \subseteq f_1 \circ g$. But $\mathcal{D}_{f_1} = \mathcal{R}_g$, so $\mathcal{D}_{f_1 \circ g} = \mathcal{D}_g = \mathcal{D}_h$. Hence, $h = f_1 \circ g$. Moreover, $f \circ g = h$ if and only if $f_1 \subseteq f$. [See, further, pages 5-89, 90, and 91, and COMMENTARY.] Exercises 3 and 5 illustrate complications which arise when $\mathcal{D}_h \subseteq \mathcal{D}_g$. In Exercise 3 the pairing of corresponding values again [as in Sample 1 and Exercise 1] gives a function f_1 such that $h = f_1 \circ g$. But now, $\mathcal{D}_{f_1} \subset \mathcal{R}_g$. In this case, $f \circ g = h$ if and only if $f_1 \subseteq f$ <u>and</u> $\mathcal{D}_f \cap \mathcal{R}_g = \mathcal{D}_{f_1}$, that is, if and only if $f_1 \subseteq f$ and those members of f which are not in f_1 have first components which are not in $\mathcal{R}_g$. In Exercise 5, although 6 and 9 are [the only] corresponding values of g and h, the function f_1 whose only member is (6, 9) is such that $h \subset f_1 \circ g$, and there is no function f such that $h = f \circ g$.

Corrections. On page 5-87, in Exercise 7, line 3 should read:

$h = \{(x, y) : y = x^2\}$

On page 5-89, in Exercise 15, line 3 should read:

h(x) = the area-measure of ...

6. $\{(x, y) : y = x^2\}$

7. $\{(x, y) : y = x^2\}$

8. Since, for example, $g(-2) = g(2)$ but $h(-2) \neq h(2)$, there is no function f such that $h = f \circ g$ [the squaring function does not have an inverse].

9. $\{(x, y), x \geq 0 : y = \sqrt{x}\}$ [$\forall_x \sqrt{x^2} = |x|$. Note that $\mathscr{D}_h = \mathscr{D}_g$ and that if $g(x_1) = g(x_2)$ then $h(x_1) = h(x_2)$.]

10. Since $0 \in \mathscr{D}_h$ but $0 \notin \mathscr{D}_g$, there is no function f such that $h = f \circ g$.

11. $\{(x, y) : xy = 1\}$ [The reciprocating function is its own inverse.]

12. $\{(7, 6), (8, 6), (9, 5)\}$

13. Since $g(3) = g(5)$ but $h(3) \neq h(5)$, there is no function f such that $h = f \circ g$.

14. $\{(1, 1), (2, 4), (3, 9), (4, 16)\}$

15. x^2; the set of nonzero numbers of arithmetic [or: the set of numbers of arithmetic]

16. $\{(0, 0), (2, 2), (5, 5), (11, 11)\}$, or any function which contains the members of this set

17. $\sqrt{x}$; the set of nonzero numbers of arithmetic [or: the set of numbers of arithmetic]

* 18. the mother of x; the set of married people

*

Note that condition (1) [above] is automatically satisfied if $\mathcal{D}_h = \mathcal{D}_g$. In this case, to test for functional dependence one merely checks to see whether h has the same value for each two arguments for which g has the same value.

Note that both condition (a) and condition (b) are automatically satisfied if g has an inverse. [See Sample 2 of Part A on page 5-93.] For, in this case, if $g(x_1) = g(x_2)$ then $x_1 = x_2$. To test for functional dependence in this case one merely checks to see whether $\mathcal{D}_h \subseteq \mathcal{D}_g$.

[Condition (a) is automatically satisfied if h has the same value for each of its arguments.]

*

The discussion on pages 5-90 through 5-91 is fairly heavy. It is probably the case that most students will accept the theorem at the bottom of page 5-91 just on the basis of the Exploration Exercises on page 5-86.

> <u>Correction.</u> On page 5-92, in line 7, change 'belong' to 'belongs'. Also, in the 8th line from the bottom of the page, change '$5 \in \mathcal{D}_h$' to '$5 \notin \mathcal{D}_h$'.

On pages 5-90 and 5-91 we develop a criterion for determining whether, given functions g and h, there is a function f such that $h = f \circ g$.

Since, for any functions f and g, $\mathcal{D}_{f \circ g} \subseteq \mathcal{D}_g$, there cannot be a function f such that $h = f \circ g$ unless $\mathcal{D}_h \subseteq \mathcal{D}_g$. [See, for example, Exercise 4 on page 5-86.] Also, if x_1 and x_2 are arguments of $f \circ g$ such that $g(x_1) = g(x_2)$ then $[f \circ g](x_1) = [f \circ g](x_2)$. So, there cannot be such a function f unless, whenever x_1 and x_2 are arguments of h such that $g(x_1) = g(x_2)$, then $h(x_1) = h(x_2)$. [See Sample 2 on page 5-86.] If this condition <u>is</u> satisfied, and $\mathcal{D}_h \subseteq \mathcal{D}_g$, then with each value of g there corresponds <u>at most</u> one value of h. In this case, the pairs $(g(x), h(x))$, for $x \in \mathcal{D}_h$, are the members of a function f_0 such that, at least, $h \subseteq f_0 \circ g$.

But, suppose that there are arguments, x_1 and x_2 of g such that $g(x_1) = g(x_2) = u$, say, and that, while $h(x_1) = v$, $x_2 \notin \mathcal{D}_h$. In such a case, $(u, v) \in f_0$, so $f_0(g(x_1)) = f_0(u) = v$ and $f_0(g(x_2)) = v$. Hence, both x_1 and x_2 belong to the domain of $f_0 \circ g$ but only one of them, x_1, belongs to $\mathcal{D}_h$. Hence, $h \neq f_0 \circ g$, since h and $f_0 \circ g$ have different domains. [See, for example, Exercise 5 on page 5-86.] So, to ensure that $h = f_0 \circ g$, rather than, merely, that $h \subseteq f_0 \circ g$, we must rule out the possibility that there are arguments x_1 and x_2 of g such that $g(x_1) = g(x_2)$, and $x_1 \in \mathcal{D}_h$, but $x_2 \notin \mathcal{D}_h$.

These three necessary conditions:

(1) $\mathcal{D}_h \subseteq \mathcal{D}_g$ [page 5-90]

(a) if $\{x_1, x_2\} \subseteq \mathcal{D}_h$ and $g(x_1) = g(x_2)$ then $h(x_1) = h(x_2)$ [page 5-91]

(b) if $\{x_1, x_2\} \subseteq \mathcal{D}_g$ and $g(x_1) = g(x_2)$ and $x_1 \in \mathcal{D}_h$ then $x_2 \in \mathcal{D}_h$ [page 5-91]

are also sufficient in order that there be a function f such that $h = f \circ g$. For, if they are satisfied then the set of pairs of corresponding values of g and h is a function f_0 such that $h = f_0 \circ g$.

8. (a) Yes. [$\mathcal{S}_A = \mathcal{S}_P$, and squares with the same perimeter have the same area-measure.]

(b) Yes. [$\mathcal{S}_P = \mathcal{S}_A$, and squares with the same area-measure have the same perimeter.]

*

Students should proceed directly to Part B after they finish Exercise 8 of Part A. If you wish to pursue the discussion on pages 5-94 and 5-95, it will be more meaningful to the students if they have pushed through the exercises in Part B.

Corrections. The second line on page 5-93 should read:

(b) Since $\mathfrak{D}_h \subseteq \mathfrak{D}_g$, we ---

On page 5-95, line 11 should read:

The converse of g is $\{(x, y): x = 3y - 4\}$,

Answers for questions in Solution for Sample 2.

p is not a function of q because $\mathfrak{D}_p \not\subseteq \mathfrak{D}_q$.

$\{(x, y): x \geq 0 \text{ and } y = 2x + 5\}$ is a function of q because its domain is $\mathfrak{D}_q$ and q has an inverse.

*

Answers for Part A [which begins on page 5-92].

1. (a) No. $[k(2) = k(4),\ \ell(2) \neq \ell(4)]$

   (b) No. $[\ell(2) = \ell(5),\ k(2) \neq k(5)]$

   (c) (2, 4) and (4, 3)

   (d) Replace (2, 4) by (2, 3); <u>or</u> replace (4, 3) by (4, 4).

2. (a) No. $[u(5) = u(7) \text{ and } 5 \in \mathfrak{D}_v \text{ but } 7 \notin \mathfrak{D}_v]$

   (b) No. $[\mathfrak{D}_u \not\subseteq \mathfrak{D}_v]$

3. Yes. [f has an inverse, and $\mathfrak{D}_g = \mathfrak{D}_f$.]

4. (a) No. $[t(1) = 3 = t(2) \text{ but } s(1) \neq s(2)]$

   (b) Yes. [s has an inverse, and $\mathfrak{D}_t = \mathfrak{D}_s$.]

5. Yes. [$\mathfrak{D}_h = \mathfrak{D}_g$, and if $g(x_1) = g(x_2)$ then $h(x_1) = h(x_2)$.]

6. (a) Yes. [$\mathfrak{D}_r = \mathfrak{D}_c$, and circles with the same circumference have the same radius.]

   (b) Yes. [$\mathfrak{D}_c = \mathfrak{D}_r$, and circles with the same radius have the same circumference.]

7. (a) No. [There are rectangles with the same perimeter but with different area-measures.]

   (b) No. [There are rectangles with the same area-measure but with different perimeters.]

Correction. At the end of the second line of the directions for Part B, the expression should be '$h \circ g^{-1}$'.

Answers for Part B.

1. $\{(8, 4), (7, 3), (6, 2), (5, 1)\}$; $\{(7, 9), (6, 14), (5, 2)\}$; $\{(3, 9), (2, 14), (1, 2)\}$

2. $\{(x, y): y = \frac{x - 9}{2}\}$; $y = 2x - 29$; $y = 4x - 11$

3. $\{(x, y): y = \frac{6 - 2x}{3}\}$; $\{(x, y), x < -3: y = 2 - 4x\}$; $\{(x, y), x > 4: y = 6x - 10\}$

4. $\{(x, y): y = x - 1\}$; $\{(x, y): y \geq 0 \text{ and } (x - 1)^2 + y^2 = 1\}$; $\{(x, y): y \geq 0 \text{ and } x^2 + y^2 = 1\}$

☆5. $\{(x, y): x \geq 0 \text{ and } y = x^2\}$; $\{(x, y): x \geq 0 \text{ and } y = x\}$; $\{(x, y), x \geq 0: y = \sqrt{x}\}$

*

A procedure very similar to that explained on page 5-94 works to find an f such that $f \circ g = h$ whenever there is such a function. [Warning. The procedure will yield a function f even when h is not a function of g. But, of course, it will not then be the case that $f \circ g = h$.] Suppose that h is a function of g and that f is the function such that $\mathcal{D}_f \subseteq \mathcal{R}_g$ and $h = f \circ g$. Now, even if g does not have an inverse there are subsets of g which have the same range as g and which do have inverses. Suppose that g_1 is such a function. That is, suppose that $g_1 \subseteq g$, $\mathcal{R}_{g_1} = \mathcal{R}_g$, and g_1 has an inverse. Then

$$h \circ g_1^{-1} = [f \circ g] \circ g_1^{-1} = f \circ [g \circ g_1^{-1}].$$

The domain of $g \circ g_1^{-1}$ is the domain of g_1^{-1}, and this is $\mathcal{R}_g$. For each $x \in \mathcal{R}_g$, $[g \circ g_1^{-1}](x) = x$. So, since $\mathcal{D}_f \subseteq \mathcal{R}_g$, the domain of $f \circ [g \circ g_1^{-1}]$ is $\mathcal{D}_f$ and, for each $x \in \mathcal{D}_f$, $[f \circ [g \circ g_1^{-1}]](x) = f([g \circ g_1^{-1}](x)) = f(x)$. Hence, $f \circ [g \circ g_1^{-1}] = f$. Consequently, $h \circ g_1^{-1} = f$. [Notice that the preceding argument is just like that on page 5-94. If g has an inverse, then $g_1 = g$. If g does not have an inverse then there will be more than one choice for g_1. Whichever choice you make, as long as $\mathcal{D}_f \subseteq \mathcal{R}_g$, $f \circ [g \circ g_1^{-1}] = f$. If $h = f \circ g$ then $h \circ g_1^{-1} = f \circ [g \circ g_1^{-1}] = f$. So, $[h \circ g_1^{-1}] \circ g = h$.

In general, the domain of $g_1^{-1} \circ g$ is $\mathfrak{D}_g$, and its range is $\mathfrak{D}_{g_1}$, a subset of $\mathfrak{D}_g$. For each $x \in \mathfrak{D}_{g_1}$, $[g_1^{-1} \circ g](x) = x$ but, in general, if $x \in \mathfrak{D}_g$, one can only say that $g([g_1^{-1} \circ g](x)) = g(x)$. If the test for functional dependence is satisfied [and only then], one can conclude that, for each $x \in \mathfrak{D}_h$, $h([g_1^{-1} \circ g](x)) = h(x)$. So, in this case, $h \circ [g_1^{-1} \circ g] = h$ and, by the associativity of function composition, $h \circ g_1^{-1}$ is a function f such that $f \circ g = h$.]

As an example, consider the functions g and h, where

$$g = \{(3, 6), (5, 9), (8, 4), (7, 6)\}$$

and

$$h = \{(3, 9), (5, 12), (8, 7), (7, 9)\}.$$

$\mathfrak{D}_h = \mathfrak{D}_g$ and 3 and 7 are the only arguments for which g has the same value. Since $h(3) = h(7)$, h is a function of g. For this function there are just two appropriate choices for g_1.

$g_1 = \{(3, 6), (5, 9), (8, 4)\}$	$g_1 = \{(5, 9), (8, 4), (7, 6)\}$
$g_1^{-1} = \{(6, 3), (9, 5), (4, 8)\}$	$g_1^{-1} = \{(9, 5), (4, 8), (6, 7)\}$
$h \circ g_1^{-1} = \{(6, 9), (9, 12), (4, 7)\}$	$h \circ g_1^{-1} = \{(9, 12), (4, 7), (6, 9)\}$

Note that with either choice, one obtains the same function $h \circ g_1^{-1}$ and that on composing this function with g one obtains h.

As another example, consider the functions g and h, where

$$g = \{(x, y): y = x^2\}$$

and

$$h = \{(x, y): x^2 \leq 1 \text{ and } y = |x|\}.$$

$\mathfrak{D}_h \subseteq \mathfrak{D}_g$ and, for two arguments x_1 and x_2 of g, $g(x_1) = g(x_2)$ if and only if x_1 and x_2 are opposites. In this case, either both or neither of them belong to $\mathfrak{D}_h$, and if both do, $h(x_1) = h(x_2)$. So, h is a function of g. A

suitable function g_1 is $\{(x, y): x \geq 0 \text{ and } y = x^2\}$. With this choice,

$$g_1^{-1} = \{(x, y): y \geq 0 \text{ and } y^2 = x\}$$

$$= \{(x, y), x \geq 0: y = \sqrt{x}\}, \text{ and}$$

$$h \circ g_1^{-1} = \{(x, y), x \geq 0: (\sqrt{x})^2 \leq 1 \text{ and } y = |\sqrt{x}|\}$$

$$= \{(x, y), x \geq 0: x \leq 1 \text{ and } y = \sqrt{x}\}.$$

Checking,

$$[h \circ g_1^{-1}] \circ g = \{(x, y), x^2 \geq 0: x^2 \leq 1 \text{ and } y = \sqrt{x^2}\}$$

$$= \{(x, y): x^2 \leq 1 \text{ and } y = |x|\}$$

$$= h.$$

An alternative choice for g_1 is $\{(x, y): x \leq 0 \text{ and } y = x^2\}$. With this choice,

$$g_1^{-1} = \{(x, y): y \leq 0 \text{ and } y^2 = x\}$$

$$= \{(x, y), x \geq 0: y = -\sqrt{x}\}, \text{ and}$$

$$h \circ g_1^{-1} = \{(x, y), x \geq 0: (-\sqrt{x})^2 \leq 1 \text{ and } y = |-\sqrt{x}|\}$$

$$= \{(x, y), x \geq 0: x \leq 1 \text{ and } y = \sqrt{x}\}.$$

So, with either choice for g_1 we obtain the same function $h \circ g_1^{-1}$.

*

Quiz.

A. $g = \{(3, 5), (2, 9), (1, 4), (0, 7)\}$
$f = \{(4, 2), (5, 1), (3, 8), (9, 11), (7, 7)\}$

1. $g^{-1} = \{$__________________$\}$ 2. $f^{-1} = \{$__________________$\}$
3. $g(3) + g^{-1}(9) =$ _____ 4. $f(4) + f(3) - f^{-1}(7) =$ _____
5. $f(g(3)) =$ _____ 6. $[f \circ g](2) =$ _____ 7. $f \circ g = \{$_____$\}$
8. $g \circ g^{-1} = \{$__________________$\}$ 9. $f \circ [g \circ g^{-1}] = \{$______________$\}$

B. 1. $a = \{(3, 8), (4, 0), (6, 5), (9, 8), (7, 9), (1, 3)\}$
$b = \{(4, 7), (6, 1), (7, 9), (1, 11)\}$
Tell why a is not a function of b.

2. $c = \{(5, 8), (4, 2), (9, 8), (10, 8)\}$ $d = \{(5, 7), (4, 9), (9, 7)\}$
Tell why d is not a function of c.

3. $e = \{(1, 2), (2, 3), (3, 4), (5, 4), (6, 3), (7, 2)\}$
$f = \{(7, 8), (6, 5), (5, 4), (3, 5), (2, 5), (1, 8)\}$
Tell why f is not a function of e.

C. Suppose that $g = \{(x, y) : y = 2x + 9\}$ and $h = \{(x, y) : y = 4x - 11\}$. Find a function f such that $h = f \circ g$.

D. Suppose that d is the function that maps each real number on its double, and t is the function that maps each real number on its triple.

1. Is t a function of d? If not, tell why. If so, describe a function f such that $t = f \circ d$.

2. Is d a function of t? If not, tell why. If so, describe a function g such that $d = g \circ t$.

*

Answers for Quiz.

A. 1. (5, 3), (9, 2), (4, 1), (7, 0)   2. (2, 4), (1, 5), (8, 3), (11, 9), (7, 7)

3. 7   4. 3   5. 1   6. 11

7. (3, 1), (2, 11), (1, 2), (0, 7)   8. (5, 5), (9, 9), (4, 4), (7, 7)

9. (5, 1), (9, 11), (4, 2), (7, 7)

B. 1. $\mathcal{D}_a \not\subseteq \mathcal{D}_b$

2. $c(9) = c(10)$ and $9 \in \mathcal{D}_d$ but $10 \notin \mathcal{D}_d$

3. $e(3) = e(5)$ but $f(3) \neq f(5)$

C. $f = h \circ g^{-1} = \{(x, y): y = 2x - 29\}$

D. 1. Yes; $f = \{(x, y): y = \frac{3}{2}x\}$ [f is the function that maps each real number on the real number $\frac{3}{2}$ times as large.]

2. Yes; $g = \{(x, y): y = \frac{2}{3}x\}$ [g is the function that maps each real number on the real number $\frac{2}{3}$ as large.]

The importance of variable quantities, and of distinguishing between variables [pronumerals] and names of variable quantities, has been stressed recently by Karl Menger [for example, in "Why Johnny Hates Math--", The Mathematics Teacher, December, 1956]. [This distinction is often lost sight of in dealing with formulas, such as '$A = s^2$', or '$v^2 = 2gh$', in which, properly speaking, 'A', 's', 'v', and 'h' are names of variable quantities, rather than pronumerals.]

Among variable quantities we include both those functions whose ranges are sets of numbers of arithmetic, and those whose ranges are sets of real numbers. [As previously, 'N' is a name for the set of numbers of arithmetic.]

*

The following exercises on pages 5-53 *et seq*. refer to variable quantities:

Part D, Exercises 3, 5, 6, 7, 9, 10, 11, 12, 15, 16 (a), (b), (c), (d), (e), 17 (a).

Part E, Exercises 1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 14.

Part F, Exercises 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14.

*

Answer to question on page 5-98, line 4.

The pair (0, 0) does not belong to the second function [but does belong to the first].

*

Remark concerning bracketed sentence on page 5-98 preceding exercises.

If f has an inverse and $A = f \circ s$ then $f^{-1} \circ A = f^{-1} \circ [f \circ s] = [f^{-1} \circ f] \circ s$.

Since $\mathcal{D}_A = \mathcal{D}_s$, $\mathcal{R}_s \subseteq \mathcal{D}_f$. So, $[f^{-1} \circ f] \circ s = s$.

Answers for Part A.

1. Yes; $v = \pi u$

2. Yes; $v = u/\pi$

*

Answers for Part B.

1. No; you can have two rectangles with the same length-measure but different area-measures.

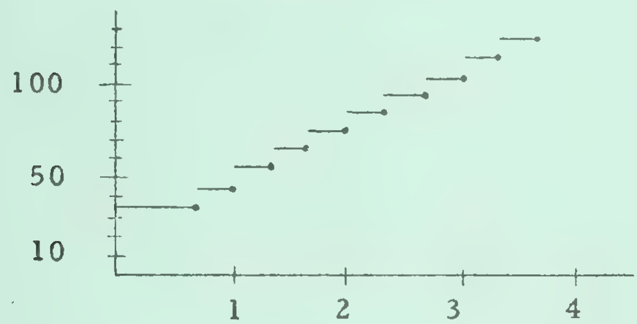
2. No; you can have two rectangles with the same area-measure but different length-measures.

*

Answers for Part C [on page 5-99].

1. Yes; here is a graph of a function g such that, for each trip x,

$$c(x) = g(m(x)).$$



2. No; trips of differing lengths may cost the same.

[Here is a description of the function g whose graph is given in the answer for Exercise 1 of Part C:

$$g(x) = \begin{cases} 35, \text{ for } 0 < x \leq 2/3 \\ 35 + 10\,(-[\![-(3x-2)]\!]), \text{ for } x > 2/3 \end{cases}$$

[The formula in the second line can, of course, be simplified to: $35 - 10[\![2 - 3x]\!]$.] See Exercise 5 of Part L of the Supplementary Exercises for a definition of '$[\![\ldots]\!]$'.]

Note concerning the word 'singulary'. The Latin distributives from which such words as 'binary', 'ternary', 'quaternary', etc. derive are 'singuli', 'bini', 'terni', 'quaterni', etc.

*

The essence of the discussion on pages 5-99 and 5-100 is that each operation on numbers can be used to induce an operation on functions. In the case of a singulary operation on numbers, the corresponding singulary operation on functions is defined in terms of the given operation on numbers just as, on page 5-100, we define the oppositing operation on functions in terms of the oppositing operation on numbers.

To bring out the fact that we are actually dealing with two operations--one on numbers and the other on functions--it might be helpful to use, at first, a bold-face '**—**' for the operation on functions. But, since one can tell from context which of the two operations is meant, there is no harm in using the same symbol, '—', for both.

The oppositing operation on functions is of little interest except as applied to functions whose values are real numbers. So, we might, in the definition on page 5-100, have replaced 'functions', both times, by 'real-valued functions', and, since each real number has an opposite, have omitted 'such that g(x) has an opposite'. Our reason for not doing so is that we want to provide a pattern for students to follow in solving Exercise 1 of Part B on page 5-101.

*

Answers for Part A [on page 5-100].

1. (a) -18 (b) 0 (c) -12 (d) -84 (e) -84

   (f) $\{(x, y): y = -(3x + 12)\}$

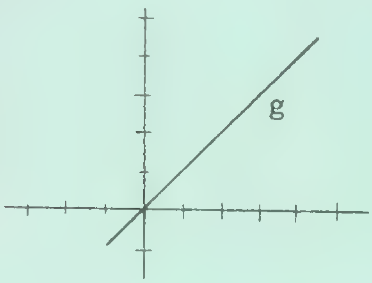
2. $\{(x, y), x \geq 0 : y = -\sqrt{x}\}$ 3. 0 4. $\frac{1}{5}$

☆5. $\{0\}$

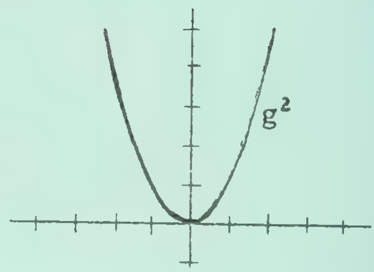
☆6. the only real number in $\mathcal{R}_f$ is 0

Answers for Part D [on page 5-101].

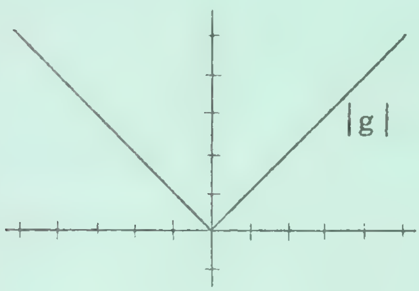
1.



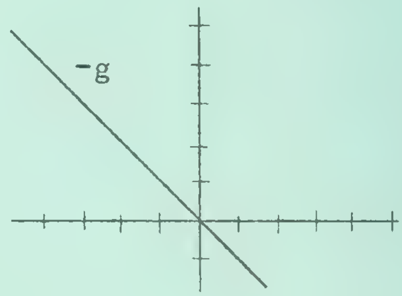
2.



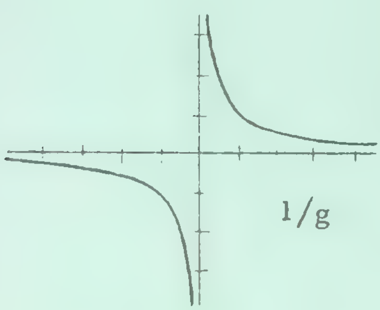
3.



4.



5.



6. $\mathcal{D}_g$ = the set of real numbers $\qquad$ $\mathcal{R}_g$ = the set of real numbers

$\mathcal{D}_{g^2}$ = the set of real numbers $\qquad$ $\mathcal{R}_{g^2} = \{x\colon\ x \geq 0\}$

$\mathcal{D}_{|g|}$ = the set of real numbers $\qquad$ $\mathcal{R}_{|g|} = \{x\colon\ x \geq 0\}$ [if '$|\ldots|$' abbreviates '$^+|\ldots|$']

$\mathcal{D}_{-g}$ = the set of real numbers $\qquad$ $\mathcal{R}_{-g}$ = the set of real numbers

$\mathcal{D}_{1/g} = \{x\colon\ x \neq 0\}$ $\qquad$ $\mathcal{R}_{1/g} = \{x\colon\ x \neq 0\}$

Students will probably follow the pattern of the definition on page 5-100, and give the answers displayed above for Exercise 1 of Part C. You might then suggest the following simpler definitions which cover all cases of interest:

(a) For each variable quantity g, g^2 is the variable quantity such that

$$g^2(x) = (g(x))^2, \text{ for each } x \in \mathfrak{D}_g.$$

(b) For each real-valued variable quantity g, $|g|$ is the variable quantity such that

$$|g|(x) = |g(x)|, \text{ for each } x \in \mathfrak{D}_g.$$

(c) For each variable quantity g, $\frac{1}{g}$ is the variable quantity such that

$$[\frac{1}{g}](x) = \frac{1}{g(x)},$$

for each $x \in \mathfrak{D}_g$ such that $g(x) \neq 0$.

2. (a) 121 (b) 100 (c) $1, -\frac{1}{7}$ (d) 46

(e) 24 (f) $4, -\frac{22}{7}$ (g) $\frac{1}{4}$ (h) $-\frac{1}{3}$

(i) $[0 \notin \mathfrak{R}_{1/g}]$ (j) $[\frac{3}{7} \notin \mathfrak{D}_{1/g}]$

(k) $\{(x, y), x \neq \frac{3}{7} : y = \frac{1}{(7x - 3)^2}\}$ [or: $\{(x, y) : y(7x - 3)^2 = 1\}$]

*

Answers for Part B.

1. $\sqrt{g(x)}$; $x \in \mathcal{D}_g$ such that $g(x)$ has a square root

2. (a) 2 (b) 14 (c) 0 (d) $[4 \notin \mathcal{D}_{\sqrt{g}}]$

   (e) $[-2 \notin \mathcal{R}_{\sqrt{g}}]$ (f) $\sqrt{g} = \{(x, y),\ x \geq 5 : y = \sqrt{x - 5}\}$

3. $\{(x, y) : y = |x|\}$

4. s

5. (a) 1 (b) 1 (c) 5 (d) $[-5 \notin \mathcal{D}_{-\sqrt{t}}]$

   (e) $[4 \notin \mathcal{D}_{\sqrt{-t}}]$ (f) $[3 \notin \mathcal{D}_{-\sqrt{t}}]$ (g) $\{(x, y),\ x \geq \frac{10}{3} : y = -\sqrt{3x - 10}\}$

   (h) $\{(x, y),\ x \leq \frac{10}{3} : y = \sqrt{-(3x - 10)}\}$

*

Answers for Part C.

1. (a) For each function g, g^2 is the function such that

$$g^2(x) = (g(x))^2,$$

for each $x \in \mathcal{D}_g$ such that $g(x)$ has a square.

(b) For each function g, $|g|$ is the function such that

$$|g|(x) = |g(x)|,$$

for each $x \in \mathcal{D}_g$ such that $g(x)$ has an absolute value.

(c) For each function g, $\frac{1}{g}$ is the function such that

$$\left[\frac{1}{g}\right](x) = \frac{1}{g(x)},$$

for each $x \in \mathcal{D}_g$ such that $g(x)$ has a reciprocal.

Instead of reinterpreting '2' as a name for a variable quantity, we might define, for each number z and each variable quantity f, $zf = \{(x, y) : y = z \cdot f(x)\}$, thus introducing a third kind of multiplication in addition to those already introduced; to wit, multiplication of numbers and multiplication of variable quantities. This procedure can be carried through, but gives rise to annoying complications. For example, in dealing with the variable quantities circumference and diameter of a circle [c and d], we want to be able to interpret not only '$c = \pi d$' [which we can do whether we consider 'π' as a numeral or as a name for a variable quantity], but also '$c/d = \pi$'. The latter makes sense only if 'π' is a name for the variable quantity c/d, i.e. for the variable quantity whose domain is the set of all circles and whose range consists of the single <u>number</u> π. With this re-interpretation of numerals as being names of variable quantities, it turns out that one can manipulate formulas according to what are, essentially, the same rules as one uses in manipulating arithmetical sentences. How this comes about is discussed on pages 5-112 through 5-115.

The ambiguity resulting from the fact that each numeral is, now, not only a name for a number but also a name for any function whose range consists of this number, does not present a serious problem. As in the case of numerals for numbers of arithmetic, which may on occasion be used as names for corresponding nonnegative real numbers, either the proper interpretation is clear from context, or it is immaterial which interpretation is adopted.

*

Note that there is some conflict between the terms 'constant variable quantity' and 'constant function' [see page 5-83]. Although a variable quantity is a function, a constant variable quantity may not be a constant function! A constant function has the set of real numbers as domain.

[The word 'constant' is also sometimes used to refer to symbols which function as nouns [for example, numerals] in contrast to symbols, called 'variables', which function as pronouns [for example, pronumerals].]

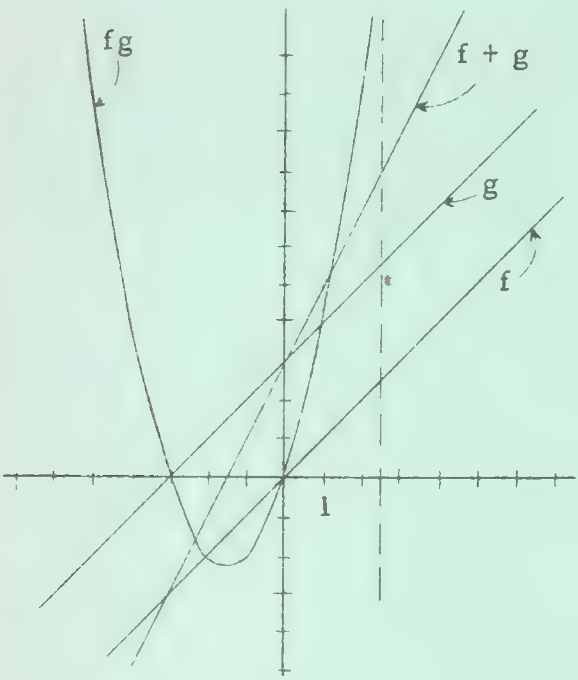
(ii) Point out that the parentheses in Exercise 3 are not punctuation, but, rather, indicate application of a function to an argument [see TC[5-55, 56]b]; and that the value of the variable quantity 3 at any of its arguments is the number 3.

(iii) Show that the incorrect answer '31' is the correct answer to another exercise: $[f^2 + 3j](2) =$ ____, where $j = \{(x, y): y = x\}$.

Similar remarks are appropriate if students give answers: 26, 51, 7, 11, 77, or: 18 for Exercises 5, 6, 7, 8, 9, 10, respectively.

Answers for Part C [on page 5-105].

1, 2, 3.



A clever way to do Exercise 2 is to lay a strip of paper vertically on the drawing [as indicated by the dotted line], mark on an edge of the strip the points where it crosses the graphs of the x-axis and g, and then shift the strip vertically until the mark which was on the graph of the x-axis is on the graph of f. Then, the other mark is on the graph of f + g. Repeat this process until you have obtained enough points on the graph of f + g. [This method of graphing is called 'addition of ordinates'. It can be a great time-saver especially in situations more complicated than that illustrated here.]

*

Answers for Part D [on page 5-105].

1. 25	2. 30	3. 28	4. 45	5. 23
6. 48	7. 6	8. 8	9. 48	10. 14

In solving Exercise 3 of Part D, students may obtain '31' as an answer rather than the correct answer: 28. In this case, apparently the closeness of the '3' and the '2' suggested multiplication. There are several ways of correcting this error.

(i) Have students show that $f^2 = \{(x, y): y = (7x - 9)^2\}$ and that $f^2 + 3 = \{(x, y): y = (7x - 9)^2 + 3\}$, so that $[f^2 + 3](2) = (7 \cdot 2 - 9)^2 + 3$.

> <u>Correction.</u> In line 5, change the exercise numeral from '2' to '3'.

Answers for Part A.

1. $h + j = \{(7, 1), (5, 2), (8, 18)\}$

2. $hj = \{(7, -12), (5, -48), (8, 65)\}$

3. $j + h = h + j$ 4. $jh = hj$

5. $7 + h = \{(3, 9), (7, 11), (0, 12), (5, 1), (8, 20)\}$

6. $3j = \{(5, 24), (8, 15), (7, -9), (2, 9), (6, 0), (9, 54)\}$

7. $h^2 = \{(3, 4), (7, 16), (0, 25), (5, 36), (8, 169)\}$

8. 1 9. −48 10. 31

11. 64 12. 23 13. 5

14. $(7, 1) \in h + j$, but $(7, 1) \notin h \cup j$ $[f + g = f \cup g \Longleftrightarrow f = g \text{ and } \mathcal{R}_f = \{0\}]$

*

Answers for Part B.

1. $c + d = \{(x, y):\ y = 5x + 2\}$

2. $cd = \{(x, y):\ y = (2x + 7)(3x - 5)\}$

3. 12 4. 170 5. 31/3 6. 31/3

[Notice, for Exercises 5 and 6, that $c + (-\frac{2}{3})d$ is a constant variable quantity.]

7. $c^2 = \{(x, y):\ y = (2x + 7)^2\}$ 8. $d^2 = \{(x, y):\ y = (3x - 5)^2\}$

*

Answers for Part E.

1. $f - g = \{(1, 2), (2, 6), (3, 4), (4, 7)\}$

2. $g - f = \{(1, -2), (2, -6), (3, -4), (4, -7)\} = -(f - g)$

3. $g/f = \{(1, 1/2), (2, 1/3), (3, 1/3), (4, 0)\}$;
   $f/g = \{(1, 2), (2, 3), (3, 3)\}$

4. $\mathcal{D}_f = \{1, 2, 3, 4\} = \mathcal{D}_g$, $\mathcal{R}_f = \{4, 6, 7, 9\}$, $\mathcal{R}_g = \{0, 2, 3\}$,
   $\mathcal{D}_{g/f} = \{1, 2, 3, 4\}$, $\mathcal{D}_{f/g} = \{1, 2, 3\}$, $\mathcal{R}_{g/f} = \{1/2, 1/3, 0\}$,
   $\mathcal{R}_{f/g} = \{2, 3\}$

*

Answers for Part F.

1. 20   2. 40   3. 0   4. 0

5. $f - 3g = \{(x, y), x > 0: y = 0\}$. It is the constant whose domain is $\{x: x > 0\}$ and whose range is $\{0\}$. It is one of the constants which we denote by '0'.

6. $\frac{1}{3}f - g = f - 3g$ 7. 3   8. 3

9. $f/g = \{(x, y), x > 0: y = 3\}$. It is the constant whose domain is $\{x: x > 0\}$ and whose range is $\{3\}$.

10. $g/f = \{(x, y), x > 0: y = 1/3\}$

*

Answers for Part G.

1. $2(x_1) = 2$; $2(x_2) = 2$ 2. $3(x_1) = 3$; $3(x_2) = 3$

3. yes, yes, yes, yes   4. $[2 + 3](x_1) = 5$; $[2 + 3](x_2) = 5$; yes

*

Quiz.

A. Suppose that M and N are variable quantities where

$$M = \{(\text{Hans}, 1), (\text{Joe}, 5), (\text{Lila}, 4), (\text{Eva}, 9)\}$$

and $N = \{(\text{Hans}, 4), (\text{Joe}, 16), (\text{Lila}, 13)\}$

1. $M(\text{Joe}) + N(\text{Joe}) =$ ____
2. $[M + N](\text{Lila}) =$ ____
3. $[-M](\text{Hans}) =$ ____
4. $[M^2](\text{Eva}) =$ ____
5. $\sqrt{N}(\text{Joe}) =$ ____
6. $MN = \{$ ____________ $\}$
7. $[M - N](\text{Joe}) =$ ____
8. $[M + N](\text{Eva}) =$ ____

B. $f = \{(x, y): y = 3x - 5\}$, $g = \{(x, y): y = 5x + 3\}$

1. $[f + g](3) =$ ____
2. $[f + g](-3) =$ ____
3. $[f - g](5) =$ ____
4. $[f - g]($__$) = 0$
5. $[fg](4) =$ ____
6. $[f \circ g](4) =$ ____
7. $[g \circ f](4) =$ ____
8. $[gf](4) =$ ____
9. $f(-4) = g($__$)$
10. $[fg](x) =$ ____
11. $[f^2](x) =$ ____
12. $[g^2](x) =$ ____
13. $[f^2 + g^2](x) =$ ____
14. $[f^2 + g](x) =$ ____
15. $[f - g^2](x) =$ ____
16. $[f/g](1) =$ ____
17. $[g/f](1) =$ ____
18. $[f/g]($__$) = 7/23$
19. $[f/g](-3/5) =$ ____
20. $[f/g]($__$) = 3/5$

*

Answers for Quiz.

A. 1. 21   2. 17   3. -1 4. 81   5. 4

6. (Hans, 4), (Joe, 80), (Lila, 52)   7. -11 8. $[\text{Eva} \notin \mathcal{D}_{M+N}]$

B. 1. 22   2. -26 3. -18 4. -4 5. 161

6. 64   7. 38   8. 161   9. -4 10. $15x^2 - 16x - 15$

11. $9x^2 - 30x + 25$ 12. $25x^2 + 30x + 9$ 13. $34x^2 + 34$

14. $9x^2 - 25x + 28$ 15. $-25x^2 - 27x - 14$ 16. $-\frac{1}{4}$

17. -4 18. 4   19. $[-\frac{3}{5} \notin \mathcal{D}_{f/g}]$ 20. $[\frac{3}{5} \notin \mathcal{R}_{f/g}]$

In reading the formula '$A = \frac{1}{2}h(b_1 + b_2)$' students should say 'one-half times aitch times bee one plus bee two'. [Since the arguments of h are not numerical, the parentheses do not indicate that h operates on $b_1 + b_2$. As is usual in formulas, the parentheses are merely grouping symbols.]

The common domain of A, $\frac{1}{2}$, h, b_1, and b_2 is the set of all trapezoids.

In the context '$V = \frac{1}{3}\pi r^2 h$', the common domain of V, $\frac{1}{3}$, π, r, h, r^2, and $\frac{1}{3}\pi$ is the set of all circular conical solids.

In the context '$c = \pi d$', the common domain of c, π, and d, is the set of all circles. And, c/d is the constant π which has this domain.

To assert '$c = \pi d$' is to say that c and πd are the same variable quantity --that is, that $\mathfrak{D}_c = \mathfrak{D}_{\pi d}$ and that, for each member x of this domain, $c(x)$ is $[\pi d](x)$.

If $d(x_1) = d(x_2)$, the circles x_1 and x_2 have the same diameter, or: are the same size, or: are congruent. Two such circles also have the same circumference--that is, $c(x_1) = c(x_2)$.

Note that the choice of which of a set of variable quantities is to be taken as the dependent one [the others, then, being called 'independent'] is largely arbitrary. For example, reference to the formula '$c = \pi d$' usually means that the speaker has chosen to consider c as the dependent variable quantity, but reference to the equivalent formula '$d = c/\pi$' indicates that he has chosen to think of d as dependent on c.

*

In the case of the rectangles x_1 and x_2, it of course does not follow from the fact that $\ell(x_1) = \ell(x_2)$ that $P(x_1) = P(x_2)$. In fact, for the given rectangles, $P(x_1) \neq P(x_2)$. And, since $\ell(x_1) = \ell(x_2)$ but $P(x_1) \neq P(x_2)$, it follows that P is <u>not</u> a function of ℓ. A similar example shows that P is not a function of w.

Two rectangles, x_3 and x_4, such that $\ell(x_3) = \ell(x_4)$ and $w(x_3) = w(x_4)$ are congruent. So, in that case, $P(x_3) = P(x_4)$.

$f((5, 2)) = 14$; $f((5, 3)) = 16$; $f((9, 6)) = 30$; $f((3, 2)) = 10$; $f((10, 7)) = 34$

*

The phrases [on page 5-109] 'functions of two variables' and 'functions of one variable' do not, of course, make literal sense. A function is not "of" anything; like any other entity, a function just "is"! But these phrases are the ones conventionally used to distinguish between functions whose arguments are ordered pairs of numbers and those whose arguments are numbers.

*

Answers for questions on page 5-109.

V is not a function of h because there exist cones x_1 and x_2 such that $h(x_1) = h(x_2)$ but $V(x_1) \neq V(x_2)$.

V is not a function of π because π is a constant variable quantity and V is not. [Similarly, V is not a function of 1/3.]

V is a function of (h, r) [as well as of (r, h)]. One function g such that $V = g((h, r))$ is

$$\{((u, v), w) \in (N \times N) \times N: \; w = \frac{1}{3}\pi uv^2\}.$$

It is different from the function f which V is of (r, h).

Answers for Part A.

1. $\mathcal{S}_K$ = {Al, Bob, Cora}

2. (a) 19 (b) 13 (c) 17 3. (a) 361 (b) 210

*

Answers for Part B [on pages 5-110 and 5-111].

1. (a) 25 (b) 13 (c) 256 (d) 840

2. (1, 55), (2, 48), (3, 27), (4, 20)

3. Yes; $\{(x, y): y = 5x - 2\}$, or: {(3, 13), (4, 18), (7, 33), (8, 38)}

4. (a) (1, 209), (2, 162), (3, 57), (4, 34)

   (b) $A = 3M^2 + 2M + 1$ ['in terms of M' means 'referring to no variable quantities other than constants and M'.]

5. (a) Yes; {(8, 64), (7, 49), (4, 16), (3, 9)} [or any more inclusive function such as $\{(x, y): y = x^2\}$]

   (b) Yes; $T = M^2$

6. (a) Yes; {(8, 15), (7, 13), (4, 7), (3, 5)} or: $\{(x, y): y = 2x - 1\}$

   (b) Yes; $S = 2M - 1$

7. (a) Yes; {(8, 98), (7, 347), (4, 629), (3, −708)} [or any more inclusive function]

   ☆(b) Yes; $Q = 63.8M^3 - 1250.95M^2 + 7733.05M - 14371.2$

There are two procedures for obtaining a function f, whose domain is the set of all real numbers, such that $Q = f \circ M$. One way begins by assuming that there are numbers a, b, c, and d such that $f(x) = a + bx + cx^2 + dx^3$, drawing the conclusion that, since $f(8) = 98$, $f(7) = 347$, $f(4) = 629$, and $f(3) = -708$,

$$a + 8b + 64c + 512d = 98,$$

$$a + 7b + 49c + 343d = 347,$$

$$a + 4b + 16c + 64d = 629,$$

and

$$a + 3b + 9c + 27d = -708.$$

It is clear that a solution (a, b, c, d) of the above equations will yield a function f which does the job. The system can be solved by standard elimination procedures, and it is seen that

$$f(x) = -14371.2 + 7733.05x - 1250.95x^2 + 63.8x^3.$$

A second procedure is to use the Lagrange interpolation formula which, for a function whose values, for four arguments x_1, x_2, x_3, and x_4, are specified to be y_1, y_2, y_3, and y_4, respectively, is:

$$f(x) = \frac{(x - x_2)(x - x_3)(x - x_4)}{(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)} y_1 + \frac{(x - x_1)(x - x_3)(x - x_4)}{(x_2 - x_1)(x_2 - x_3)(x_2 - x_4)} y_2$$

$$+ \frac{(x - x_1)(x - x_2)(x - x_4)}{(x_3 - x_1)(x_3 - x_2)(x_3 - x_4)} y_3 + \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_4 - x_1)(x_4 - x_2)(x_4 - x_3)} y_4$$

In the present case, this yields:

$$f(x) = \frac{(x - 7)(x - 4)(x - 3)}{(8 - 7)(8 - 4)(8 - 3)} \cdot 98 + \frac{(x - 8)(x - 4)(x - 3)}{(7 - 8)(7 - 4)(7 - 3)} \cdot 347$$

$$+ \frac{(x - 8)(x - 7)(x - 3)}{(4 - 8)(4 - 7)(4 - 3)} \cdot 629 + \frac{(x - 8)(x - 7)(x - 4)}{(3 - 8)(3 - 7)(3 - 4)} \cdot -708$$

To see how this works, note that, according to the above formula,

$$f(8) = 1 \cdot 98 + 0 \cdot 347 + 0 \cdot 629 + 0 \cdot -708 = 98,$$

$$f(7) = 0 \cdot 98 + 1 \cdot 347 + 0 \cdot 629 + 0 \cdot -708 = 347,$$

$$f(4) = 0 \cdot 98 + 0 \cdot 347 + 1 \cdot 629 + 0 \cdot -708 = 347,$$

and $$f(3) = 0 \cdot 98 + 0 \cdot 347 + 0 \cdot 629 + 1 \cdot -708 = -708.$$

[There are, of course, many other functions which can be used in place of f.]

*

8. (a) $\{(x, y)\colon\ y = 3x + 5\}$, or: $\{(3, 14), (4, 17), (7, 26), (8, 29)\}$
   (b) Yes; $\{(x, y)\colon\ y = (x - 5)/3\}$
   (c) $M = (R - 5)/3$

Correction. On page 5-112, the last part of line 17 should be 'quantity as fh + gh'.

9. (a) Yes; $\{(x, y): y = x - 5\}$; $E = M - 5$

(b) Yes; $M = E + 5$

(c) $(1, 3/8)$, $(2, 2/7)$, $(3, -1/4)$, $(4, -2/3)$

(d) Yes; $\{(x, y): xy = x - 5\}$, or: $\{(x, y), x \neq 0 : y = 1 - 5/x\}$

(e) $E/M = 1 - 5/M$

10. (a) $B = M$ (b) $C = 3$, where 3 is the variable quantity $\{(1, 3), (2, 3), (3, 3), (4, 3)\}$.

☆11. (a) That constant 2 whose domain is $\{1, 2, 3, 4\}$ is a function of M. [A constant is a function of each variable quantity which has the same domain.]

(b) No; $[2(1) = 2(2)$, but $M(1) \neq M(2)]$

*

Answers for Part C.

1. (a) 43 (b) 51 (c) −21 (d) 29 (e) 1849 (f) 836

2. (a) $(1, -1)$, $(2, -1)$, $(3, 23)$, $(4, -23)$

(b) $(1, -96)$, $(2, -133)$, $(3, -85)$, $(4, -144)$

(c) $(1, -96)$, $(2, -133)$, $(3, -85)$, $(4, -144)$

*

Answers for Part D.

1. $A(x_2) = A(x_1) + 15$

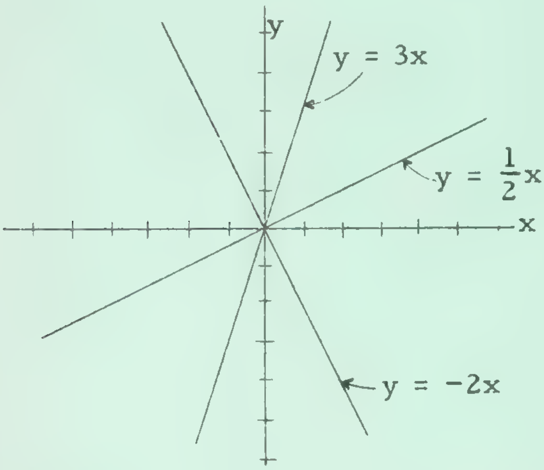
2. $A(x_2) = 7 \cdot A(x_1)$

*

Answers for Exploration Exercises.

Answers for Part A [on page 5-115].

1.



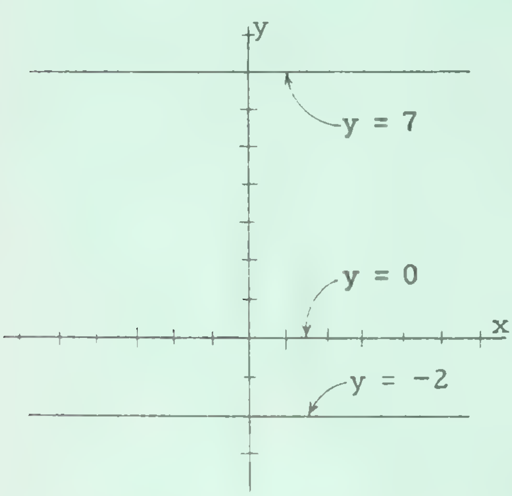
2. $\{(0, 0)\}$

3. $\{(0, 0)\}$

*

Answers for Part B [on page 5-115].

1.



2. ϕ

3. ϕ

*

Answers for Quiz.

A. 1. $(e_1, 13), (e_2, 10), (e_3, 7), (e_4, 16)$

2. $(e_1, -1), (e_2, -1), (e_3, -1), (e_4, -1)$

3. $(e_1, 97), (e_2, 58), (e_3, 29), (e_4, 146)$

4. Yes; $f = \{(x, y): y = 2x + 1\}$ [or any function which contains as a subset $\{(4, 9), (3, 7), (2, 5), (5, 11)\}$]

B. 1. 65 2. 51 3. 79 4. 66 5. 4225 6. 4356

7. Yes; $f = \{(x, y): y = 7x + 2\}$ [or any function which contains as a subset $\{(9, 65), (7, 51), (5, 37), (11, 79)\}$]

8. Yes, $g = \{(x, y): y = 14x + 9\}$ [or any function which contains as a subset $\{(4, 65), (3, 51), (2, 37), (5, 79)\}$]

9. $M = 14A + 9$ [since $M = 7B + 2$ and $B = 2A + 1$]

C. 1. 100π 2. 84π 3. 2 4. 1/4 5. 1/9

*

Quiz.

A. Suppose that A and B are variable quantities where

$$A = \{(e_1, 4), (e_2, 3), (e_3, 2), (e_4, 5)\}$$

and $B = \{(e_1, 9), (e_2, 7), (e_3, 5), (e_4, 11)\}$.

1. If $C = A + B$ then $C = \{$______________________$\}$.
2. If $D = 2A - B$ then $D = \{$______________________$\}$.
3. If $E = A^2 + B^2$ then $E = \{$______________________$\}$.
4. Is B a function of A? If not, tell why. If so, describe a function f such that $B = f \circ A$.

B. Suppose that A and B are the variable quantities described in Part A, and that M is the variable quantity $7B + 2$.

1. $M(e_1) =$ _____ 2. $M(e_2) =$ _____ 3. $M(e_4) =$ _____
4. $[M + 1](e_1) =$ _____ 5. $[M^2](e_1) =$ _____
6. $[M^2 + 2M + 1](e_1) =$ _____
7. Is M a function of B? If not, tell why. If so, describe a function f such that $M = f \circ B$.
8. Is M a function of A? If not, tell why. If so, describe a function g such that $M = g \circ A$.
9. Write a formula for M in terms of A.

C. The formula '$V = \frac{1}{3}\pi hr^2$' tells you that the volume of a circular cone is $\frac{1}{3}\pi$ times the height of the cone times the square of the radius of the base.

1. If $h(c_1) = 12$ and $r(c_1) = 5$ then $V(c_1) =$ _____.
2. If $h(c_2) = 9/7$ and $r(c_2) = 14$ then $V(c_2) =$ _____.
3. If $r(c_3) = r(c_4)$ and $h(c_3) = 2[h(c_4)]$ then $V(c_3) \div V(c_4) =$ _____.
4. If $h(c_5) = h(c_6)$ and $r(c_5) = 1/2[r(c_6)]$ then $V(c_5) \div V(c_6) =$ _____.
5. If $V(c_7) = V(c_8)$ and $r(c_7) = 3[r(c_8)]$ then $h(c_7) \div h(c_8) =$ _____.

Answers for Exercises.

1. $b = P - a - c$

2. $a = \frac{P - b}{2}$

3. $h = \frac{3V}{B}$

4. $c = 2s - a - b$

5. $h = \frac{3V}{\pi r^2}$

6. $b_1 = 2m - b_2$

7. $b_1 = \frac{2A}{h} - b_2$

8. $a = \frac{2S}{n} - \ell$

9. $n = \frac{2S}{a + \ell}$

10. $h = \frac{T - 2\pi r^2}{2\pi r}$

11. $r = \frac{E}{C} - R$

12. $r = 1 - \frac{a}{S}$

13. $a = \frac{S(r - 1)}{r^n - 1}$

14. $V = \frac{KT}{P}$

15. $d = \frac{rS}{2\pi r^2 - S}$

16. $M = \frac{6V - (B_1 + B_2)h}{4h}$

*

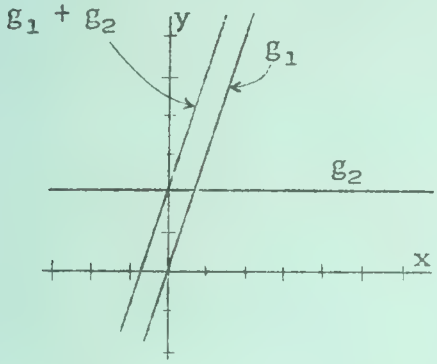
There is another analogy between operations on functions and operations on numbers which you may wish to point out to your students. Consider all functions whose arguments and values are real numbers. When one adds or composes such functions he obtains functions of the same kind--that is, this set of functions is closed under the operations of addition and composition. As we have seen, addition of functions is commutative and associative, and function composition is associative, but not commutative. Moreover, it is easy to see that function composition is distributive with respect to addition in the sense that the generalization:

$$\forall_f \forall_g \forall_h \, [f + g] \circ h = f \circ h + g \circ h$$

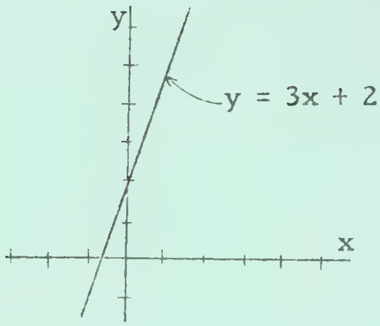
holds. [That a left-hand distributive principle does not hold is shown by, among other examples, the well-known fact that the square of a sum is not the sum of the squares.] If one restricts one's consideration to real-valued functions whose domain is the set of all real numbers then addition of functions is a commutative group operation [see TC[5-264]]. In this case, including composition gives an example of an important kind of mathematical system called a ring. Since, for each f, $f \circ 1 = f = 1 \circ f$, it is a ring with unit.

Answers for Part C.

1.



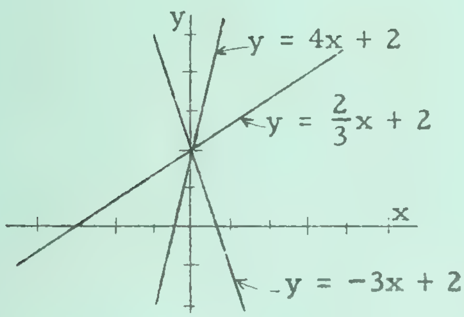
2.



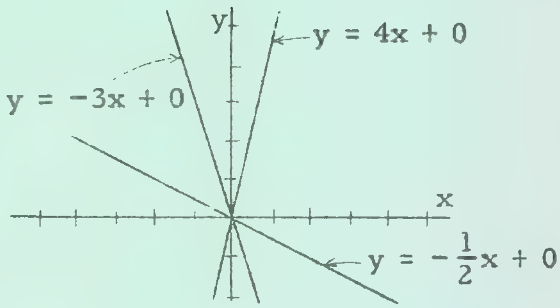
*

Answers for Part D.

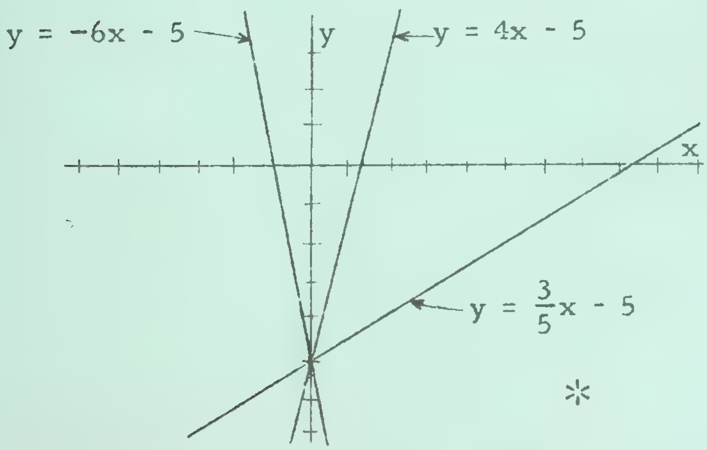
1.



2.



3.



*

Answers for Part E [on page 5-116].

[Students should be allowed to use whatever algebraic techniques they can discover in doing these problems.]

1. $\{(0, 5)\}$ 2. $\{(0, -7)\}$ 3. $\{(0, 9)\}$ 4. $\{(0, 0)\}$ 5. $\emptyset$ 6. $\emptyset$

*

Answers for Part F [on pages 5-116 and 5-117].

1. 7	2. 83	3. 12	4. 0	5. 5	6. 52
7. 1/5	8. 2	9. 1/2	10. 5/2	11. $-1/2$	

IV. 1. (a) $\{1, 3, 5\}$ (b) $\{2, 4, 6\}$ (c) $\{1, 2, 3, 4, 5, 6\}$

(d) $\{(2, 1), (4, 3), (6, 5)\}$

2. (a) the set of real numbers (b) the set of nonnegative real numbers

(c) the set of real numbers (d) $\{(x, y) : y = \sqrt{x} \text{ or } y = -\sqrt{x}\}$

V. 1. (a) $\frac{(x - 3)}{4}$ (b) $2(x - 5)$ (c) 2 (d) 2 (e) $\frac{39}{8}$ (f) $\frac{-9}{4}$

2. (a) the set of all real numbers (b) $\{x: x \geq 1\}$

(c) 13 (d) 13 (e) 169 (f) -13 (g) 1 (h) 1

(i) (13, 2) and (13, -2) both belong to the converse of f; so, the converse of f is not a function.

(j) $\{(x, y), x \geq 0: y = 3x^2 + 1\}$ [There are many others.]

3. (a) (1, 5), (2, 6), (3, 6), (4, 4) (b) (5, 2), (6, 1), (4, 2)

(c) 2 (d) 2 (e) 2 and 3 (f) (1, 2), (2, 1), (3, 1), (4, 2)

VI. 1. $f = \{(x, y) : y = x^2\}$ 2. $f = \{(x, y) : y = \frac{1}{2}x + 4\}$

3. No; $g(1) = 3 = g(2)$ but $h(1) \neq h(2)$.

VII. 1. Union of two straight lines, one containing the points (0, 8) and (8, 0) and the other containing the points (0, 0) and (1, 3). Both lines contain the point (2, 6).

2. The set consisting of just the point (2, 6).

3. The union of the first and third quadrants.

4. The circular region with radius 3 and center (0, 0) including the boundary.

5. A right angle with vertex (3, 0) and symmetric with respect to the straight line containing (3, 0) and parallel to the second component axis. The angle opens upward.

6. A parabola opening upward with extreme point (-3, 0) and containing the points (0, 9) and (-6, 9).

VI. For each pair of functions, g and h, tell whether h is a function of g. If not, tell why. If so, describe a function f such that $h = f \circ g$.

1. $g = \{(x, y) : y = |x|\}$, $h = \{(x, y) : y = x^2\}$
2. $g = \{(x, y) : y = 2x\}$, $h = \{(x, y) : y = x + 4\}$
3. $g = \{(x, y) : y = 3\}$, $h = \{(x, y) : y = 3 - x\}$

VII. Graph each of the relations listed below.

1. $\{(x, y) : x + y = 8\} \cup \{(x, y) : y = 3x\}$
2. $\{(x, y) : x + y = 8\} \cap \{(x, y) : y = 3x\}$
3. $\{(x, y) : xy > 0\}$
4. $\{(x, y) : x^2 + y^2 \leq 9\}$
5. $\{(x, y) : y = |x - 3|\}$
6. $\{(x, y) : y = (x + 3)^2\}$

*

Answers for Quiz.

I.

1. T	2. F	3. T	4. T	5. T	6. T	7. F
8. T	9. T	10. F	11. T	12. T	13. F	14. T
15. T	16. T	17. F	18. F	19. T	20. T	21. F
22. F	23. T	24. T	25. F			

II. 1. (1, 1) (3, 3), (5, 5), (7, 7), (9, 9), (11, 11), (13, 13)

2. (1, 1), (2, 2), (3, 3), (4, 4), (5, 5)

3. (9, 9), (9, 10), (9, 11), (10, 9), (10, 10), (10, 11), (11, 9), (11, 10), (11, 11)

4. $6 < x < 14$ 5. $x = 110$ 6. $\{2, 4, 6, 8\}$ [or: $\mathcal{R}_K$]

7. 4 8. 1/3 9. $-2x + 8$

10. quadrupling function [or: 4-times function, or: the function that maps each real number on the real number 4 times as large]

III. 1. C 2. S and C

3. (a) 4 (b) 21 (c) 9 (d) 16 (e) 21 (f) 21 (g) 4
(h) 4 (i) 12 (j) 9 (k) 0 (ℓ) 25 (m) 13 (n) 4

4. (c) 5. (b)

IV. For each of the two relations given below, tell its

(a) domain (b) range (c) field (d) converse

1. $H = \{(1, 2), (3, 4), (5, 6)\}$ 2. $J = \{(x, y) : y = x^2\}$

V. 1. Suppose that $f = \{(x, y) : y = 4x + 3\}$ and $g(x) = \frac{1}{2}x + 5$.

(a) $f^{-1} = \{(x, y) : y =$ ________$\}$ (b) $g^{-1}(x) =$ ________

(c) $f^{-1}(f(2)) =$ ________ (d) $[f \circ f^{-1}](2) =$ ________

(e) $g(f^{-1}(2)) =$ ________ (f) $[f^{-1} \circ g^{-1}](2) =$ ________

2. Suppose that $f = \{(x, y) : y = 3x^2 + 1\}$

(a) $\mathcal{D}_f =$ ________ (b) $\mathcal{R}_f =$ ________

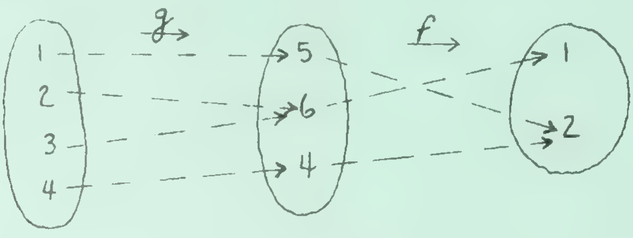
(c) $f(2) =$ ______ (d) $f(-2) =$ ______ (e) $[f^2](2) =$ ______

(f) $[-f](2) =$ ____ (g) $\sqrt{f}(0) =$ ______ (h) $[1/f](0) =$ ______

(i) Prove that f does not have an inverse.

(j) Describe a subset of f that does have an inverse.

3. The mappings g and f are shown below.

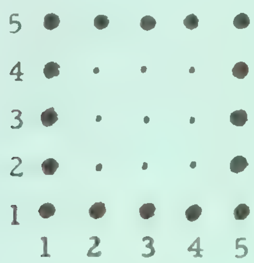


(a) $g = \{$________________$\}$ (b) $f = \{$________________$\}$

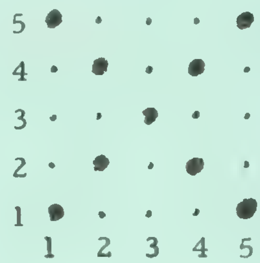
(c) $f(g(1)) =$ ____ (d) $f(g(4)) =$ ____ (e) $f(g($__$)) = 1$

(f) $f \circ g = \{$________________$\}$

III. Suppose V = {1, 2, 3, 4, 5} and S and C are relations among the members of V. Here are graphs of S and C.



<u>Graph of S</u>



<u>Graph of C</u>

1. Which of these relations are reflexive?
2. Which of these relations are symmetric?
3. How many elements are in each of the sets listed below? [Complements are with respect to $V \times V$.]

   (a) $S \cap C$ (b) $S \cup C$ (c) $\tilde{S}$ (d) $\tilde{C}$

   (e) $\tilde{S} \cup \tilde{C}$ (f) $\widetilde{S \cap C}$ (g) $\tilde{S} \cap \tilde{C}$ (h) $\widetilde{S \cup C}$

   (i) $S \cap (\tilde{S} \cup \tilde{C})$ (j) $C \cup (S \cap C)$ (k) $S \cap (\tilde{S} \cap \tilde{C})$

   (ℓ) $S \cup (\tilde{S} \cup \tilde{C})$ (m) $(S \cap C) \cup \tilde{S}$ (n) $(S \cup C) \cap (S \cap C)$

4. Which of the following describes S?

   (a) $\{(x, y) \in V \times V: (x < 2 \text{ or } x > 4) \text{ and } (y < 2 \text{ or } y > 4)\}$

   (b) $\{(x, y) \in V \times V: |x - 3| > 1 \text{ and } |y - 3| > 1\}$

   (c) $\{(x, y) \in V \times V: x = 1 \text{ or } x = 5 \text{ or } y = 1 \text{ or } y = 5\}$

   (d) $\{(x, y) \in V \times V: 1 < x < 5 \text{ and } 1 < y < 5\}$

   (e) $\{(x, y) \in V \times V: 1 < x < 5 \text{ or } 1 < y < 5\}$

5. Which of the following describes C?

   (a) $\{(x, y) \in V \times V: x = y \text{ or } x = -y\}$

   (b) $\{(x, y) \in V \times V: |x - 3| = |y - 3|\}$

   (c) $\{(x, y) \in V \times V: |x| = |y|\}$

   (d) $\{(x, y) \in V \times V: x - 3 = y - 3\}$

   (e) $\{(x, y) \in V \times V: 3 - y = x - 3\}$

22. For all variable quantities M and N, the domain of M + N is the union of the domains of M and of N.

23. For each real-valued function f, the range of the function $-f$ consists of the opposites of the values of f.

24. For each real-valued function f and for each argument x of f such that $f(x) \geq 0$, $\sqrt{f}(x) = \sqrt{f(x)}$.

25. If $a \in \mathcal{D}_R$ but $(a, b) \notin R$, it follows that $b \notin \mathcal{R}_R$.

II. Fill the blanks.

1. The reflexive relation whose field is $\{1, 3, 5, 7, 9, 11, 13\}$ and which has the fewest members is $\{$__________________$\}$.

2. The intersection of all reflexive relations whose field is $\{1, 2, 3, 4, 5\}$ is $\{$____________________$\}$.

3. The union of all reflexive relations whose field is $\{9, 10, 11\}$ is $\{$_______________$\}$.

4. The three sides of a triangle are 4 inches, 10 inches and x inches long if and only if _______________.

5. The degree-measures of the angles of a triangle are 20, 50 and x if and only if _________________.

6. If K is a symmetric relation and $\mathcal{R}_K = \{2, 4, 6, 8\}$ then $\mathcal{D}_K =$ ____________________.

7. (5, ____) $\in \{(x, y) \in I^+ \times I^+ : 4x - 2 < y + 15 < 3x + 5\}$

8. If $(3, 8) \in \{(x, y) : y = ax + 7\}$ then a = ______.

9. If $f = \{(x, y) : x + 2y = 8\}$ then $f^{-1} = \{(x, y) : y =$ ______$\}$.

10. If d is the doubling function [that is, the function that maps each real number on its double] then $d \circ d$ is the _________.

Examination [for pages 5-A through 5-115].

I. True or false?

1. A variable quantity is a numerical valued function.
2. Every relation is a function.
3. Every function is a relation.
4. Intersecting is distributive with respect to unioning.
5. For all subsets x and y of a given set S, $x \cup (x \cap y) = x$.
6. For all subsets x and y of a given set S, $y \cap (x \cup y) = y$.
7. $\forall_x\ x + 3 = 5$
8. $\exists_x\ x + 7 = 12$
9. $\forall_x\ (\exists_y\ x + y = 10)$
10. $\exists_x\ (\forall_y\ x + y = 10)$
11. There is a unique value corresponding to each argument of a function.
12. For each function f and for all arguments x_1 and x_2 of f, if $f(x_1) \neq f(x_2)$ then $x_1 \neq x_2$.
13. The relation $\{(x, y) : y < x\}$ is symmetric.
14. The relation $\{(x, y) : y \geq x\}$ is reflexive.
15. The converse of a symmetric relation is symmetric.
16. The converse of a reflexive relation is reflexive.
17. For all relations R_1 and R_2, if $R_1 \cup R_2$ is symmetric and R_1 is symmetric then R_2 is symmetric.
18. The perimeter of a rectangle is a function of its length-measure.
19. The area-measure of an equilateral triangle is a function of its side-measure.
20. The degree-measure of an angle of a triangle is a function of the sum of the degree-measures of the other two angles.
21. The measure of a side of a triangle is a function of the sum of the measures of the other two sides.

12. 2

13. 3

14. 2 , 5

15. 2 , $\boxed{3}$

16. 2 , $\boxed{7}$

17. 2 , $\boxed{77}$

18. 2 , 17

19. 2 , $\boxed{7}$

20. 2 , $\boxed{4}$

21. 2 , $\boxed{-1}$

22. 8 , $\boxed{2}$

23. −4 , $\boxed{3}$

24. 5 , 12

25. 3 , 21

26. 329 , $\boxed{210}$

27. 4 , $\boxed{13}$, 26

28. 4 , $\boxed{-13}$, −26

*

Answers for Part G [on pages 5-117 and 5-118].

1. 19k 2. −29k 3. ak 4. m/k 5. b 6. b, 7 + b

7. b, 21 + b 8. b, a + b, 2a + b 9. b, ap

10. 9 + b, 9p 11. 14 + b, −2p 12. a3 + b, ap

13. aq + b, ap 14. ap

Note that a constant function of a real variable is a constant--that is, a variable quantity whose range is a singleton. But, of course, not every constant is a constant function of a real variable.

*

Answer for question in the last paragraph on page 5-118.

The domain, and the range, of each of the given functions is the set of real numbers.

*

Answers for Part A [on pages 5-120 and 5-121].

1. Yes; $a = -5$, $b = 3$
2. Yes; $a = 3/2$, $b = 5/2$
3. Yes; $a = 1/3$, $b = -7/3$
4. Yes; $a = 5$, $b = 11$
5. Yes; $a = -1$, $b = 9$
6. No; $\mathcal{D}_f = \{0, 1, 2\} \neq$ the set of real numbers [However, f is a subset of the linear function $\{(x, y) : y = x + 9\}$.]

If the graphs of Figure 1 and Figure 2 [on page 5-119] were drawn on the same chart then that of Figure 3 could be formed by the method of addition of ordinates. [See discussion on TC[5-105]b.]

*

Answers for questions in first paragraph on page 5-119.

For the value 0 of 'a', and each value of 'b', $\{(x, y): y = ax + b\}$ is a constant function, and its graph is a horizontal line.

None of the graphs are vertical lines. [After all, the sets are functions.]

For each nonzero value of 'a', and each value of 'b', the graph is an oblique straight line.

*

The explanation called for in the last paragraph on page 5-119 might run as follows: In the preceding paragraph of the text, linear functions have been identified as functions whose graphs are oblique straight lines. Now, $\{(x, y): y = 7\}$ is a function whose graph is a straight line, but not an oblique one; and $\{(x, y): x = 4\}$ is not even a function [although its graph is a straight line]. Hence, neither of these sets of ordered pairs is a linear function.

Another interesting function to consider is the function g defined by:

$$g(x) = \frac{x^2 - 4}{x - 2}$$

Its graph is an oblique straight line [$y = x + 2$] with a hole at (2, 4). It is not a linear function because its domain is not the set of all real numbers.

*

Answers for questions on page 5-120.

If $f(x) = ax + b$, then $f(0) = b$ and $f(1) = a + b$. Hence, if $a \neq 0$, $f(0) \neq f(1)$. Consequently, no linear function is a constant function.

The domain of each linear function is the set of all real numbers.

The domain of $\{(x, y): y = 7x - 5 \text{ and } |y| \leq 1000000\}$ is not the set of real numbers. For example, 1000000 does not belong to the domain. [The largest member of the domain is $(10^6 + 5)/7$, and the smallest is $(-10^6 + 5)/7$.] So, the function in question is not a linear function.

*

$\{(x, y): C = 0\}$, and is either the cartesian square of the set of real numbers, or $\emptyset$, according as $C = 0$ or $C \neq 0$. So, if $B = 0$ then $\{(x, y): Ax + By + C = 0\}$ is not a function unless, also, $A = 0$ and $C \neq 0$. In this case it is not a linear function.

Having treated all cases, we see that $\{(x, y): Ax + By + C = 0\}$ is a linear function if and only if neither A nor B is 0.

*

Answer to question at foot of page 5-122.

$(3, -5) \notin g$ because g, being a linear function, has an inverse, and $(0, -5) \in g$.

☆3. If f and g are linear functions then there are numbers $a \neq 0$ and b such that $f = \{(x, y): y = ax + b\}$, and numbers $c \neq 0$ and d such that $g = \{(x, y): y = cx + d\}$. So, $f \circ g = \{(x, y): y = (ac)x + (ad + b)\}$. Since $ac \neq 0$, $f \circ g$ is a linear function. [Note that if either f or g is a linear function and the other is a constant function then $f \circ g$ is a constant function.]

*

Answers for Part C [on page 5-121].

1. Yes
2. No [The product of each function whose domain is the set of real numbers by the constant function 0 is also this constant function and, so, no such product is a linear function.]
3. No [The sum of the linear functions $\{(x, y): y = x\}$ and $\{(x, y): y = -x\}$ is the constant function 0.]

[In extension of Exercises 2 and 3 of Part C, note that the product of a linear function by any constant function other than 0 is a linear function; and that the sum of a linear function and a linear function is either a linear function or a constant function.]

*

Answers for Part ☆D [on page 5-121].

1. Yes [All such relations for which $AB \neq 0$.]
2. constant functions; no functions; linear functions containing (0, 0); the function $\emptyset$; not a function, just the set of all ordered pairs of real numbers
3. Suppose $B \neq 0$. Then

$$\{(x, y): Ax + By + C = 0\} = \{(x, y): y = (-A/B)x - C/B\}.$$

If, in addition, $A \neq 0$ then $-A/B \neq 0$. Hence, if neither A nor B is 0, the function in question is a linear function with $a = -A/B$ and $b = -C/B$. Also, if $A = 0$ and $B \neq 0$, the function is a constant function and so is <u>not</u> a linear function.

It remains to be shown that if $B = 0$ then $\{(x, y): Ax + By + C = 0\}$ is not a linear function. Now, if $A \neq 0$, this set of ordered pairs is $\{(x, y): x = -C/A\}$, and is not a function. And, if $A = 0$, it is

Corrections. Exercises 2 and 3 of Part B should be starred.

On page 5-122, in the last line, insert a ']' after 'isn't?'.

7. Yes; $a = 14/29$, $b = -37/29$

8. No; $f = \{(x, y): x = 3\}$, and this is not a function.

9. No; f is a constant function.

10. Yes; $a = 7$, $b = 0$

11. No; $\mathfrak{D}_f = \{x: x \geq 0\} \neq$ the set of real numbers

12. Yes; $a = -1$, $b = 0$

13. No; $\mathfrak{D}_f = I \neq$ the set of real numbers

14. No; f is not a function.

15. Yes; $a = -2$, $b = 11$

16. Yes; $a = -5/3$, $b = 5$

17. Yes; $a = -5/7$, $b = 31/7$

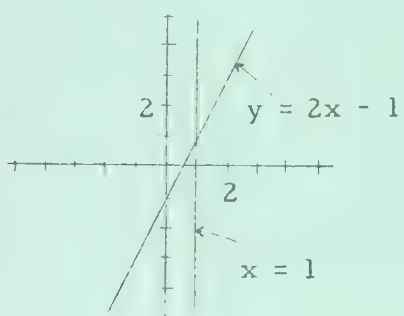
18. Yes; $a = -2/3$, $b = 13/3$

19. Yes; $a = -3$, $b = 0$

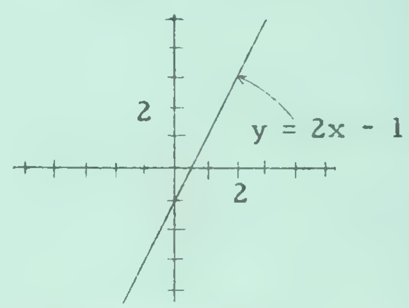
20. No; f is not a function.

21. No; $\mathfrak{D}_f = \{0\} \neq$ the set of real numbers

☆22. No; f is not a function.



☆23. Yes; $a = 2$, $b = -1$ [Note that $\forall_x\ x^2 + 1 \neq 0$.]



[Try: $f = \{(x, y): x^2 - 2xy + y^2 = 0\}$ as an extra exercise. This is, of course, just $\{(x, y): y = x\}$.]

*

Answers for Part B.

1. Yes [$f^{-1} = \{(x, y): y = (x + 5)/2\}$]; yes [$a = 1/2$, $b = 5/2$]

☆2. If f is a linear function then there are numbers $a \neq 0$ and b such that $f = \{(x, y): y = ax + b\}$. The converse of f is $\{(x, y): x = ay + b\}$ -- that is, since $a \neq 0$, $\{(x, y): y = (1/a)x - (b/a)\}$. So, if f is a linear function then so is its converse. Hence, each linear function has an inverse, and this inverse is a linear function.

Quiz.

Each of the relations described below is a linear function. For each of them, write the defining equation in the form 'y = ax + b'.

Sample. $\{(x, y): 2x - 3y - 7 = 0\}$; $y = \_\_\_ x + \_\_\_$

Solution. $y = \underline{2/3}\, x + \underline{-7/3}$

1. $\{(x, y): 5x - y = 10\}$; $y = \_\_\_ x + \_\_\_$

2. $\{(x, y): 16x = 3 - 4y\}$; $y = \_\_\_ x + \_\_\_$

3. the function which maps each real number on its double; $y = \_\_\_ x + \_\_\_$

4. the function which maps each real number on its opposite; $y = \_\_\_ x + \_\_\_$

5. the sum of $\{(x, y): x = 2y - 5\}$ and $\{(x, y): y = 2x - 5\}$; $y = \_\_\_ x + \_\_\_$

6. the composition of $\{(x, y): x = 2y - 5\}$ with $\{(x, y): y = 2x - 5\}$; $y = \_\_\_ x + \_\_\_$

7. the inverse of $\{(x, y): 3x + 4y + 2 = 0\}$; $y = \_\_\_ x + \_\_\_$

*

Answers for Quiz.

1. $5, -10$
2. $-4, \frac{3}{4}$
3. $2, 0$
4. $-1, 0$
5. $\frac{5}{2}, -\frac{5}{2}$
6. $1, 0$
7. $-\frac{4}{3}, -\frac{2}{3}$

Notice the inequations in lines 15 and 16 of page 5-125. You can, when going over this material in class, insert a very brief review of procedures for transforming inequations [See Unit 3, page 3-100.]:

$$\begin{aligned} &ak > 0 \\ &ak + q > 0 + q \quad \Big\} \text{ atpi} \\ &q + ak > q \quad \Big\} \text{ cpa, pa0} \end{aligned}$$

*

Answers to questions on page 5-125, lines 19ff.

If a is positive and k is negative, the graph of (p + k, q + ak) is below and to the left of the graph of (p, q). Yes, the graph of the function rises to the right.

When a is negative and k is either positive or negative, the graph rises to the left. If $a < 0$ and $k < 0$ then $ak > 0$. So, $q + ak > q$.

When the intercept is positive, the graph of f crosses the vertical axis above the horizontal axis; when the intercept is negative, the graph crosses below the horizontal axis; when the intercept is 0, the graph crosses the horizontal axis at the graph of (0, 0).

*

Answers to questions at top of page 5-126.

The graph rises to the right if the slope is positive, to the left if the slope is negative. No, the slope cannot be 0.

A valuable kind of classroom exercise consists in drawing numerous oblique lines on the chalk board and asking students to estimate the slopes of the linear functions whose graphs they are. Disputes should be resolved by taking rough measurements [perhaps using the eraser length as unit], and taking account of the direction of rise.

Notice that, although we have chosen to use pronumerals 'a' and 'b' in discussing linear functions, it would have been very natural to have introduced the variable quantities <u>the slope of a linear function</u> and <u>the intercept of a linear function</u>. Had we done so, and used 'a' and 'b' as names for these variable quantities, then, in place of '$y = ax + b$' as the type of defining equation for a linear function, we should have used '$y = a(f) \cdot x + b(f)$'. Here, instead of the numerical variables 'a' and 'b' we have the linear functional variable 'f'.

Answers for Part A [on pages 5-127 and 5-128].

(1) $\frac{2}{3}$, -2, $y = \frac{2}{3}x - 2$

(2) $-\frac{2}{3}$, -2, $y = -\frac{2}{3}x - 2$

(3) 2, 0, $y = 2x$

(4) $\frac{3}{2}$, -4, $y = \frac{3}{2}x - 4$

(5) $-\frac{1}{4}$, 1, $y = -\frac{1}{4}x + 1$

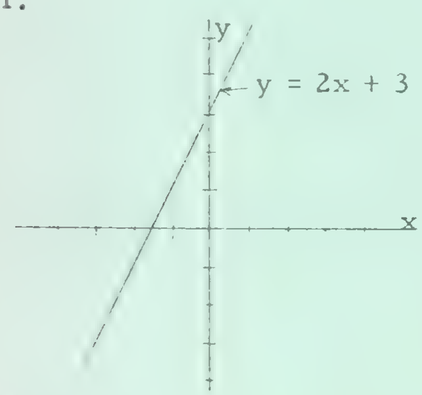
(6) 2, 1, $y = 2x + 1$

In finding the intercept of the function graphed in Exercise 6, students may argue, as on page 5-125, that, since $a = 2$, if $k = 2$ then $ak = 4$, and the intercept, b, is $-3 + 4$.

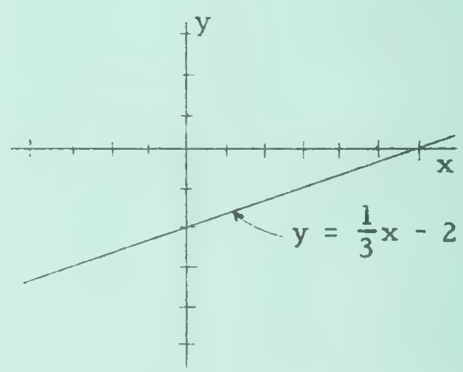
*

Answers for Part B [on page 5-128].

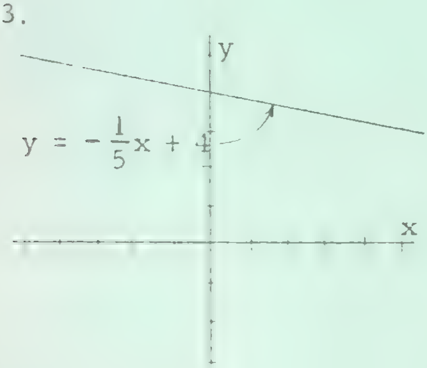
1.



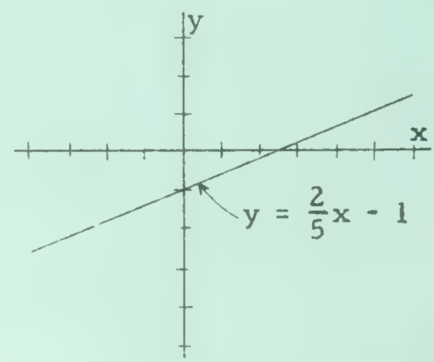
2.



3.



4.



*

Correction. On page 5-128, Exercise 3 of Part D should be starred.

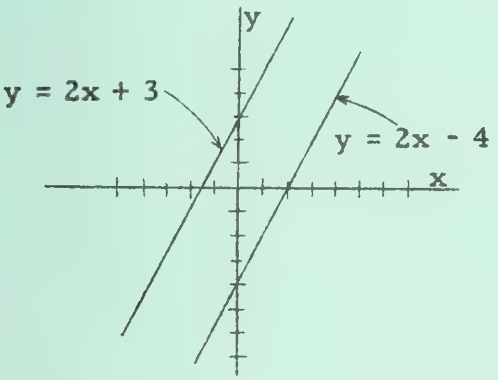
Answers for Part C.

No. We could define the slope of a constant function to be 0. The intersection of each constant function, $\{(x, y): y = b\}$, and $\{(x, y): x = 0\}$ is $\{(0, b)\}$. We could define the intercept of a constant function to be its value.

*

Answers for Part D.

1.



The intersection of the two functions is $\emptyset$.

☆ 2. As in the hint, if $(p, q) \in f_1 \cap f_2$ then $(a_1 - a_2)p = b_2 - b_1$. Now, if f_1 and f_2 have the same slope then $a_1 = a_2$, and [since $0 \cdot p = 0$] it follows that $b_1 = b_2$. Hence $f_1 = f_2$. [We have shown that if f_1 and f_2 are linear functions with the same slope and a nonempty intersection, then $f_1 = f_2$.] So, if $f_1 \neq f_2$ [that is, if f_1 and f_2 are <u>two</u> linear functions] and f_1 has the same slope as f_2, then there is <u>no</u> $(p, q) \in f_1 \cap f_2$; that is, $f_1 \cap f_2 = \emptyset$.

☆ 3. If f_1 is f_2 then f_1 and f_2 have the same slope, but $f_1 \cap f_2 \neq \emptyset$.

Answers for Part E.

1. 2   2. $\frac{4}{3}$ 3. $\frac{3}{4}$ 4. $-\frac{7}{5}$ 5. -1 6. -1 7. $-\frac{4}{13}$

8. No linear function contains both points.

9. No function [linear or otherwise] contains both points.

☆10. $\frac{y_2 - y_1}{x_2 - x_1}$

*

Answers for Part F.

These exercises foreshadow the discussion starting on page 5-134.

1. 2   2. 2   3. 2

4. Yes $[\{(x, y): y = 2x\}]$. This illustrates the fact that if the linear function containing a first point and a second point has the same slope as the linear function containing the second point and a third point then they are the same linear function.

   No. [A graph shows this quite readily, and, obviously, there are no numbers $a \neq 0$ and b such that $2 = 1a + b$ and $2 = 2a + b$. Or, since a linear function has an inverse, (1, 2) and (2, 2) cannot belong to the same linear function.] This illustrates the fact that the linear function containing two points may differ from the linear function containing two other points even though the functions have the same slope.

*

Explanation asked for on page 5-130, line 8.

There are as many linear functions containing (5, 7) as there are nonzero real numbers because there are as many such functions as there are possible slopes.

*

A quick way to obtain a defining equation for any linear function which contains (5, 7) is to substitute for 'a' in 'y - 7 = a(x - 5)'.

The displayed principle in the second paragraph on page 5-131 is analogous to that given in Exercise 6 on page 2-66 of Unit 2. Here is a proof.

Suppose that	$x = y$ and $u = v$.
Since	$x - u = x - u$,
it follows that	$x - u = y - u$,
and, hence, that	$x - u = y - v$.
Therefore,	if $x = y$ and $u = v$ then $x - u = y - u$.

Notice that no mathematical principles are involved in the proof. Only logical rules and logical principles are needed. [Compare with the solution for Exercise 6 on TC[2-66]b.]

*

[One purpose of the exercises in Parts G and H is to introduce students to methods of solving two linear equations in two variables [See pages 5-200 ff.]. So, students should use the procedure given in the Sample.]

Answers for Part G [on page 5-132].

1. $\{(x, y): y = \frac{1}{2}x - \frac{3}{2}\}$ 2. $\{(x, y): y = \frac{3}{4}x\}$ 3. $\{(x, y): y = -x + 5\}$
4. $\{(x, y): y = 3x + 5\}$ 5. $\{(x, y): y = -5x - 5\}$ 6. $\{(x, y): y = 12x\}$
7. No linear function contains both ordered pairs.
8. $\{(x, y): y = -x + 5\}$ 9. $\{(x, y): y = -\frac{3}{4}x + 3\}$
10. $\{(x, y): y = \frac{3}{4}x + 3\}$ 11. <u>No</u> function contains both (3, 5) and (3, 7).

☆12. If $x_1 \neq x_2$ and $y_1 \neq y_2$ then $\{(x, y): y = \frac{y_1 - y_2}{x_1 - x_2} \cdot x + (y_1 - x_1 \cdot \frac{y_1 - y_2}{x_1 - x_2})\}$ is a linear function which contains both ordered pairs.
[Simpler descriptions are:

$$\{(x, y): y = \frac{y_1 - y_2}{x_1 - x_2} \cdot x + \frac{x_1 y_2 - x_2 y_1}{x_1 - x_2}\},$$

and:

$$\{(x, y): \frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}\}.]$$

If $x_1 = x_2$ and $y_1 = y_2$ [so that, contrary to the instructions, there are not two ordered pairs listed] then, for each $a \neq 0$, $\{(x, y): y = ax + (y_1 - ax_1)\}$ is a linear function which contains the given ordered pair. In any other case [$x_1 = x_2$ and $y_1 \neq y_2$, or $x_1 \neq x_2$ and $y_1 = y_2$], either no function or, at least no linear function contains both ordered pairs.

Actually, the slope of the linear function which contains the points (0, 0) and (2, 5) is less than that of the linear function which contains (2, 5) and (5, 13). So, the rectangle is not completely covered by the pieces of the square. The part not covered is [enclosed by] a parallelogram, one of whose diagonals is a diagonal of the rectangle, and whose area-measure is 1.

*

Answers for Part J [on page 5-133].

1. Yes [$\{(x, y): y = x\}$]
2. Yes [$\{(x, y): y = 2x + 2\}$]
3. Yes [$\{(x, y): y = \frac{1}{2}x + \frac{17}{2}\}$]
4. Yes [$\{(x, y): y = -x + 6\}$]
5. No
6. Yes [$\{(x, y): y = -\frac{1}{5}x + 1\}$]
7. No
8. No

*

<u>Quiz.</u>

1. What are the slope and the intercept of $\{(x, y): 2x - 4y = 5\}$?
2. Write the defining equation of the linear function which contains the points (3, 8) and (−2, 4). [y = ____x + ____]
3. Find the ordered pair which belongs to the functions $\{(x, y): 4x + y = 7\}$ and $\{(x, y): y = 2x - 5\}$.
4. Write the defining equation of the linear function whose slope is 7 and which contains the point (−2, 8).
5. Find two ordered pairs which belong to the inverse of the linear function which contains (0, 8) and (8, 0).
6. Write a brace-notation name for the constant function which contains the point (−3, −2).

*

Answers for Quiz.

1. $\frac{1}{2}$; $-\frac{5}{4}$
2. $y = \frac{4}{5}x + \frac{28}{5}$
3. (2, −1)
4. $y = 7x + 22$
5. (0, 8) and (8, 0) [or any other two points in the linear function $\{(x, y): y + x = 8\}$]
6. $\{(x, y): y = -2\}$

Answers for Part H [which begins on page 5-132].

1. $\{(10, 38)\}$ 2. $\{(-1, -4)\}$ 3. $\{(18, 2)\}$ 4. $\emptyset$

5. $\{(-3, -3)\}$ 6. $\{(1, 1)\}$ 7. $\{(1, 5)\}$ 8. $\{(6, 2)\}$

9. $\{(-3, 9)\}$ 10. $\emptyset$ 11. $\{(x, y): y = -\frac{1}{2}x + \frac{9}{2}\}$

[Students are expected to invent techniques for Exercises 6 - 11.]

*

Answers for Part I.

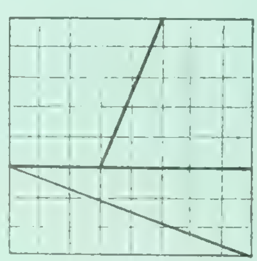
1. $\{(x, y): y = \frac{4}{3}x + \frac{13}{3}\}$ 2. There is no such linear function.

3. The slope of the linear function containing $(-89, -93)$ and $(67, -8)$ is $85/156$, and that of the linear function containing $(67, -8)$ and $(15, -37)$ is $87/156$. So, no linear function contains all three ordered pairs. However, each of the sets of three ordered pairs listed below is a subset of some linear function.

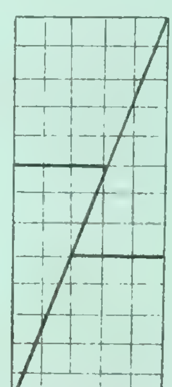
$$\{(-89, -93), (67, -8), (15, -109/3)\}$$
$$\{(-89, -95), (67, -8), (15, -37)\}$$
$$\{(-89, -93), (67, -9), (15, -37)\}$$
$$\{(-89, -93), (67, -8), (1171/85, -37)\}$$
$$\{(-89, -93), (482/7, -8), (15, -37)\}$$
$$\{(-89, -2269/29), (67, -8), (15, -37)\}$$

*

Here is a familiar puzzle whose solution is like that of Exercise 3 of Part I. Apparently, a square can be cut into 4 pieces which, when reassembled, make up a rectangle whose area-measure is greater than that of the original square. [Solution on next page.]



$8 \times 8 = 64$



$5 \times 13 = 65$

Corrections. In the last line at the bottom of page 5-135, insert an 'if' after 'only'.

In the first line on page 5-136, insert a '[' in front of 'suppose'.

Answer to bracketed question at top of page 5-136: No.

[Ask students to compare Exercises 2 and 10 of Part G on page 5-132. If the pairs given in Exercise 2 are used for (x_4, y_4) and (x_5, y_5) respectively, the slope is the same as if the pairs given in Exercise 10 are used. Yet, the pairs given in Exercise 10 do not belong to the same linear function as do the pairs given in Exercise 2. Also, recall Exercise 4 at bottom of page 5-129.]

Answers for Part K [on pages 5-137 and 5-138].

1. Yes 2. Yes 3. No 4. No 5. Yes 6. No

*

Answers for Part L [on page 5-138].

Explanation: If g and h are linear functions then their common domain is the set of all real numbers; so, $\mathcal{D}_h = \mathcal{D}_g$. Also, by Exercise 2 of Part B on page 5-121, g has an inverse, and g^{-1} is a linear function. By Exercise 3 [same Part], since both h and g^{-1} are linear functions, so is $h \circ g^{-1}$.

1. $f = \{(x, y): y = -2x - 9\}$
2. $f = \{(x, y): 9x + 4y = 13\}$
3. $f = \{(x, y): x = 15y + 2\}$
4. $f = \{(x, y): 2x + 15y = 23\}$

Here are two ways of solving Exercise 3, above. [Each of the other three exercises can be solved in the same two ways.]

Since $g = \{(x, y): y = 3x - 4\}$, $g^{-1} = \{(x, y): y = \frac{x + 4}{3}\}$.

Hence, $h \circ g^{-1} = \{(x, y): \frac{x + 4}{3} = 5y + 2\}$.

This function, $\{(x, y): x = 15y + 2\}$, is a linear function f such that $h = f \circ g$.

An alternate method of solution makes use of the fact that, since g and h are linear functions, there is a linear function f such that $h = f \circ g$. That is, there are numbers $a \neq 0$ and b such that, for each x, $h(x) = a \cdot g(x) + b$; that is, such that

$$\forall_x \ \frac{x}{5} - \frac{2}{5} = a(3x - 4) + b.$$

By inspection, $a = \frac{1}{15}$ and, since, in that case, $-\frac{2}{5} = -\frac{4}{15} + b$, it follows that $b = -\frac{2}{15}$. So,

$$f = \{(x, y): y = \frac{1}{15}x - \frac{2}{15}\} = \{(x, y): x = 15y + 2\}.$$

Correction. On page 5-139, part (c) of Exercise 2 should be starred.

5. If f has an inverse then, for each $x \in \mathcal{D}_f$, $(f(x), x) \in f^{-1}$. So, for each $x \in \mathcal{D}_f$, $[f^{-1} \circ f](x) = f^{-1}(f(x)) = x$. In particular, $\mathcal{D}_f \subseteq \mathcal{D}_{f^{-1} \circ f}$ and since, for each function g, $\mathcal{D}_{g \circ f} \subseteq \mathcal{D}_f$, $\mathcal{D}_{f^{-1} \circ f} = \mathcal{D}_f$. Summarizing, for each $x \in \mathcal{D}_{f^{-1} \circ f}$, $[f^{-1} \circ f](x) = x$. So, $f^{-1} \circ f = \{(x, y) \in \mathcal{D}_f \times \mathcal{D}_f : y = x\}$. Now, if g is a real-valued function, f is a linear function, and $h = f \circ g$, then $f^{-1} \circ h = f^{-1} \circ [f \circ g] = [f^{-1} \circ f] \circ g = g$, by the result just proved. Since f^{-1} is a linear function, g is a linear function of h. [Essentially, we have proved a more general result: If f has an inverse, $h = f \circ g$, and $\mathcal{R}_g \subseteq \mathcal{D}_f$, then $g = f^{-1} \circ h$.]

*

The remark at the foot of page 5-138 is placed there because of its application to Exercises 1 and 2 of Part M on page 5-139. In general, measures are numbers of arithmetic and to be able to say that variable quantities whose values are measures are linear functions of one another, one must confuse numbers of arithmetic with the corresponding nonnegative real numbers. We shall consistently do this in the next section, which begins on page 5-140. The justification for doing so is discussed in the COMMENTARY for Unit 3 on TC[3-55]. Briefly, doing so simplifies the algebraic manipulations [one does not need to restrict subtraction to the case where the minuend exceeds the subtrahend]; and, the fact that the system of numbers of arithmetic is isomorphic, with respect to addition and multiplication, to the system of nonnegative real numbers, ensures that the results obtained by this procedure can be properly interpreted.]

*

Answers for Part M [on page 5-139].

1. (a) Yes (b) $\{(x, y) : y = \pi x\}$

2. (a) No (b) ___ ☆(c) Yes; the linear function in question is $\{(x, y) : y = x\}$

3. (a) $A = 2B - 1$ (b) $A = 5B - 1$ (c) $A = (B - 1)/3$
   (d) $A = -B$ (e) $A = (B - 6)/5$ (f) $A = (B - 6)/5$
   (g) $A = -B + 9$ (h) $A = B^2$ (i) $A = 24/B$

*

Quiz.

1. Is the perimeter (P) of an equilateral triangle a linear function of its side-measure (s)? If not, tell why. If so, describe the linear function f such that $P = f \circ s$.

2. Suppose that A and B are variable quantities where

$$A = \{(e_1, 5), (e_2, 6), (e_3, 7), (e_4, 8)\}$$

and $B = \{(e_1, 9), (e_2, 12), (e_3, 15), (e_4, 18)\}$.

Write a formula for B in terms of A.

3. If f is a linear function which contains $(-2, 5)$ and $(8, 7)$ then f also contains $(13, \underline{\ ?\ })$.

4. Describe the linear function r such that $p = r \circ q$ where
$q = \{(x, y): x + 2y = 6\}$ and $p = \{(x, y): 3x - y = 5\}$.

*

Answers for Quiz.

1. Yes; $f = \{(x, y): y = 3x\}$

2. $B = 3A - 6$

3. 8

4. $r = \{(x, y): y = -6x + 13\}$

Answers for True-False questions on pages 5-140 and 5-141.

1. T 2. F 3. T [if 'temperature' refers to absolute temperature; otherwise, not.]

4. T 5. F 6. T 7. F [if he has a fixed charge for making a call, in addition to a charge depending on time worked.]

8. F 9. T 10. F

Some students may feel that one variable quantity is proportional to another if they "increase together"--that is, if "when one increases, so does the other". The only answer to such feelings is to say that people who use words properly don't use this word so loosely. The matter should be cleared up by the definition on page 5-141, so, don't spend more time than absolutely necessary discussing questions 1 - 10.

*

Notice that, in the definition of proportionality, 'k' is a pronumeral, and the factor of proportionality is a number. This is consistent with the equation '$P(e) = kQ(e)$'. On the other hand, if one writes '$P(e) = [kQ](e)$' or, as we later do, '$P = kQ$', then, in each of these equations 'k' names the constant variable quantity whose domain is the common domain of P and Q and whose range consists of the factor of proportionality.

*

The perimeter of a rectangle whose length-measure is 5 is given by the formula:

$$P = 10 + 2w$$

If the width-measure of such a rectangle is 1 then its perimeter is 12; if the width-measure is 2, the perimeter is 14. Since, if $12 = k \cdot 1$ then $k = 12$, and if $14 = k \cdot 2$ then $k = 7$, and, since $12 \neq 7$, there is no number k such that, for each rectangle r, $P(r) = k \cdot w(r)$. So, the perimeter of a rectangle whose length-measure is 5 is not proportional to its width-measure. [However, P is proportional to $w + 5$. Compare with True-False question 3. The volume, V, of a gas sample is not proportional to its centigrade temperature, t; but, V is proportional to $t + 273$. In general, if a variable quantity P is a linear function of some variable quantity Q then P is proportional to the sum of Q and some constant.]

*

Answers for Part A [on page 5-142].

1. Yes; 3 2. No 3. Yes; -3 4. No

5. Yes; 5/3 6. Yes; 1 7. Yes; 3/8 8. No

Answers for Part B.

1. $c = \{(x, y) \in \text{Circles} \times N\colon\ y \text{ is the circumference of } x\}$,
   $r = \{(x, y) \in \text{Circles} \times N\colon\ y \text{ is the radius of } x\}$; so, $\mathcal{D}_c = \mathcal{D}_r$.

   From geometry, we know that there is a number $2\pi \neq 0$ such that, for each $e \in \mathcal{D}_r$, $c(e) = 2\pi \cdot r(e)$. Hence, the circumference of a circle is proportional to its radius, and the factor of proportionality is 2π.

2. Yes. [The factor of proportionality is 6.]

3. Let Q be $\{(e, s) \in \text{Squares} \times N\colon\ s \text{ is the side-measure of } e\}$, and P be $\{(e, A) \in \text{Squares} \times N\colon\ A \text{ is the area-measure of } e\}$.

   Suppose that e_1 is a square whose side-measure is 1, and e_2 is a square whose side-measure is 2. If P <u>is</u> proportional to Q, then there must exist a number $k \neq 0$ such that $P(e_1) = kQ(e_1)$ and $P(e_2) = kQ(e_2)$. If this is so, then,

   since $P(e_1) = 1$ and $Q(e_1) = 1$, $1 = k \cdot 1$ and $k = 1$,

   and, since $P(e_2) = 4$ and $Q(e_2) = 2$, $4 = k \cdot 2$ and $k = 2$.

   Since $1 \neq 2$, it follows that there exists no number $k \neq 0$ such that, for each $e \in \mathcal{D}_Q$, $P(e) = kQ(e)$.

4. $V = \{(x, y) \in \text{Cones} \times N\colon\ x \text{ is a circular cone whose base has radius-measure 3, and } y \text{ is the volume-measure of } x\}$,

   $h = \{(x, y) \in \text{Cones} \times N\colon\ x \text{ is a circular cone whose base has radius-measure 3, and } y \text{ is the height-measure of } x\}$;

   so, $\mathcal{D}_V = \mathcal{D}_h$.

   From geometry, we know that there is a number $3\pi \neq 0$ such that, for each $e \in \mathcal{D}_h$,

   $$V(e) = \frac{1}{3}\pi \cdot 3^2 \cdot h(e) = 3\pi \cdot h(e).$$

   Hence, V is proportional to h, with factor of proportionality 3π.

*

Answers for Part C.

1. Yes 2. No 3. Yes 4. No 5. No 6. Yes

Correction. On page 5-145, in line 4 of Exercise 19, change 'm + q' to 'm + p'.

Answers for Part D.

1. is [$\frac{1}{k}$] 2. is [slope = factor of proportionality, intercept = 0]

3. k; $\frac{1}{k}$ 4. (a) k (b) Yes, a constant variable quantity

*

Answers for Part E [on pages 5-144 and 5-145].

1. 4 2. 1.2 3. 14 4. 30 5. 7 6. 9 7. 2

8. If, for some $e \in \mathcal{D}_Y$, $Y(e) = 0$, then, for this e, $X(e) = \frac{0}{k} = 0$. But, [by assumption] 0 is not a value of X. Hence, there is no $e \in \mathcal{D}_Y$ such that $Y(e) = 0$--that is, 0 is not a value of Y.

[An alternative proof. We are given that Y is proportional to X, and that no value of X is 0. So, there exists a $k \neq 0$ such that,

$$\text{for each } e \in \mathcal{D}_X, \quad Y(e) = kX(e).$$

By the 0-product theorem, if $k \neq 0$ and $X(e) \neq 0$ then $kX(e) \neq 0$. Hence, $Y(e) \neq 0$.]

9. 19 10. $\frac{y_1}{x_1} = k = \frac{y_2}{x_2}$ 11. $\frac{x_1}{y_1} = \frac{1}{k} = \frac{x_2}{y_2}$ 12. 3

13. By Exercise 11, $\frac{x_1}{y_1} = \frac{x_2}{y_2}$. So, [by the MTP, etc.] $x_1y_2 = x_2y_1$. ["Product of the means equals product of the extremes."]

14. $\frac{3}{2}$ 15. 9 16. $\frac{17}{3}$

17. By Exercise 10, $\frac{y_1}{x_1} = \frac{y_2}{x_2}$. 18. $\frac{3}{8}$

So, $\frac{y_1}{x_1} + 1 = \frac{y_2}{x_2} + 1$. [ATP]

Hence, $\frac{y_1 + x_1}{x_1} = \frac{y_2 + x_2}{x_2}$.

☆19. (a) $y_1 = kx_1$ and $y_2 = kx_2$

So, $y_1 + y_2 = kx_1 + kx_2$,

that is, $y_1 + y_2 = k(x_1 + x_2)$.

Hence, $\dfrac{y_1 + y_2}{x_1 + x_2} = k$.

But, $\dfrac{y_1}{x_1} = k$.

Therefore, $\dfrac{y_1 + y_2}{x_1 + x_2} = \dfrac{y_1}{x_1}$.

(b) Suppose that, for some $a \neq 0$, (m, n) and (p, q) belong to $\{(x, y): y = ax\}$. Then,

$n = am$ and $q = ap$.

So, $n + q = am + ap$

and $n + q = a(m + p)$.

Hence, $(m + p, n + q) \in \{(x, y): y = ax\}$.

*

On page 5-146, the answer to the bracketed question preceding 'EXERCISES' is 'Yes'.

*

Answers for Part A [on pages 5-146 and 5-147].

1. 15	2. $\frac{27}{7}$	3. 10	4. 8
5. $\frac{45}{2}$	6. 40	7. $\frac{4}{15}$	8. $\frac{15}{7}$
9. 0.75	10. 6, −6	11. 4, −4	12. 8, −8

*

Quiz.

1. Suppose that P and Q are variable quantities such that

$$P = \{(e_1, 3), (e_2, 5), (e_3, 7), (e_4, 9)\}$$

and $Q = \{(e_1, 6), (e_2, 9), (e_3, 12), (e_4, 15)\}$.

(a) Is Q a linear function of P? If not, tell why.
If so, describe the linear function f such that $Q = f \circ P$.

(b) Is Q proportional to P? If not, tell why. If so, give the factor of proportionaltiy.

2. Suppose that A is proportional to B and that (a_1, b_1) and (a_2, b_2) are ordered pairs of corresponding values of A and B. If $a_1 = 7$, $b_1 = 12$, and $a_2 = 3$, what is b_2?

3. Prove that the area-measure (A) of a rectangle is not proportional to its length-measure (ℓ).

*

Answers for Quiz.

1. (a) Yes; $f = \{(x, y) : y = \frac{3}{2}x + \frac{3}{2}\}$

(b) No; $Q(e_1) = 2 \cdot P(e_1)$, $Q(e_2) = \frac{9}{5} \cdot P(e_2)$, and $2 \neq \frac{9}{5}$. [So, there is no number $k \neq 0$ such that for each $e \in \mathcal{D}_P$, $Q(e) = kP(e)$.]

2. 36/7

3. For each rectangle r, $A(r) = w(r) \cdot \ell(r)$. Consider the rectangles r_1 and r_2 where r_1 is a 3 by 5 rectangle and r_2 is a 4 by 10 rectangle. Then, $A(r_1) = 3 \cdot \ell(r_1)$ and $A(r_2) = 4 \cdot \ell(r_2)$, and $3 \neq 4$.

13. $\frac{29}{4}$ 14. $-\frac{5}{2}$ 15. 10

16. (4, 9) [You might remind students of the convention mentioned in Unit 4 regarding the solution of sentences which contain the pronumerals 'x' and 'y'. It is customary, when giving solutions to sentences involving more than one variable, to list the components of the solution in the order which corresponds to the alphabetical order of the variables.]

17. (2, 10) 18. $(\frac{32}{3}, 3, \frac{20}{3})$

*

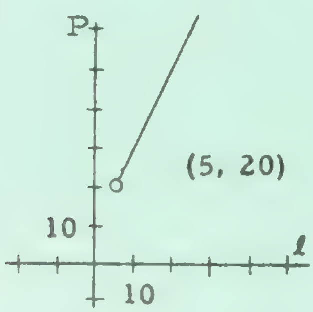
Answers for Part B [on pages 5-147, 5-148, 5-149].

1. $\frac{P_1}{s_1} = \frac{P_2}{s_2}$, or: $\frac{P_1}{P_2} = \frac{s_1}{s_2}$, or: $\frac{s_1}{P_1} = \frac{s_2}{P_2}$, or: $\frac{P_2}{P_1} = \frac{s_2}{s_1}$

2. A = 5B

3. Yes; the factor of proportionality is $2\sqrt{2}$, so '2.8', or even '3' would be an acceptable answer.

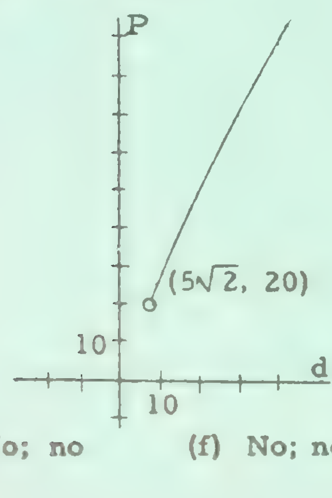
4. (a)



The appropriate formula for part (a) is, of course, '$P = 10 + 2\ell$', and the graphs of the ordered pairs found by students should fit somewhere on the half-line in the accompanying sketch.

(b) Yes (c) No

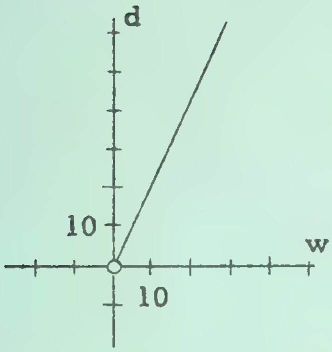
(d)



The appropriate formula for part (d) is '$P = 10 + 2\sqrt{d^2 - 25}$' and the graphs of the ordered pairs found by students should fit somewhere on the piece of a hyperbola shown in the accompanying sketch. Students' graphs are very likely to suggest that P is a linear function of d, and to leave it somewhat in doubt whether P is proportional to d. In fact, P is not a linear function of d [see formula above], and so, in particular, is not proportional to d. This can be shown by graphing the students' data, using a larger scale on the horizontal axis than on the vertical axis.

(e) No; no (f) No; no

5. (a)



Since $\ell = 2w$, and $d = \sqrt{\ell^2 + w^2}$, the appropriate formula for part (a) is '$d = \sqrt{5}w$'. The graphs of students' ordered pairs should fit somewhere on the half-line shown in the accompanying sketch.

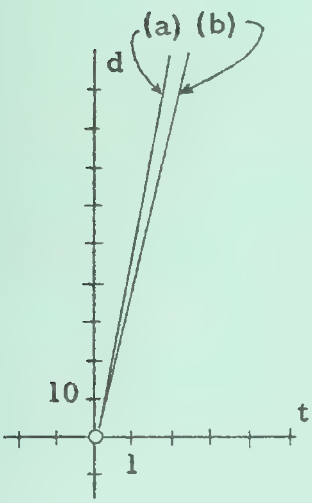
(b) Yes; yes

(c) Yes; yes; [6]

(d) Yes; yes; [$\frac{6}{\sqrt{5}}$, approximately $\frac{8}{3}$]

(e) No

6.



The appropriate formulas for parts (a) and (b) are, of course, '$d = 50t$' and '$d = 40t$', respectively.

[Note that the axes are graphed with different scales.]

(c) In the case of the more rapidly moving car, the graph is steeper.

(d) 260 miles

(e) Yes; in part (a), the factor of proportionality is 50, and in part (b), 40.

*

Answers for Part C [on page 5-149].

1. $3 + \frac{2}{B}$; not proportional

2. 5; proportional; 5

[Note that in the answer for Exercise 1, '3' and '2' name constants. In the answer for Exercise 2, the first '5' names a constant, but the second '5' names a number.]

3. π; proportional; π

4. πx; not proportional

5. 9×10^{16}; proportional; 9×10^{16}

6. $\frac{3}{Q^2}$; not proportional

[Notice that if M and N are variable quantities, with the same domain,

whose ratio is a nonzero constant k, then, <u>if 0 is not a value of N</u>, M is proportional to N. For, in this case, it follows from the principle of quotients that M = kN. But, if N(e), say, is 0, the formula '$\frac{M}{N} = k$' tells us nothing about M(e). In order, in this case, to be sure that M is proportional to N we must also know that M has the value 0 for each argument for which N has the value 0.]

*

Quiz.

1. Solve these proportions.

   (a) $\frac{x}{15} = \frac{9}{5}$ (b) $\frac{12}{x - 2} = \frac{4}{7}$ (c) $\frac{2}{x} = \frac{x}{18}$

2. For each of the following formulas, tell whether A is proportional to B. [Assume that all other letters in the formulas stand for nonzero constants.]

   (a) $A = kB$ (b) $A = g^2B$ (c) $A = kB + m$

   (d) $B = sA$ (e) $A = k\sqrt{B}$ (f) $p^2A = q^2B$

*

Answers for Quiz.

1. (a) 27 (b) 23 (c) 6, −6

2. (a) Yes (b) Yes (c) No (d) Yes (e) No (f) Yes

Correction. In line 19
. . . a physics or chemistry test-

Students should see that, since $20.65 < 22.37$ and $1.0852 > 1.0020$, $20.65/1.0852 < 22.37/1.0020$. So, they should see, without computing, that the ratios for the first two pairs of values listed in the table are different.

*

When completed, the table at the bottom of page 5-151 should look like the following:

b(r)	24	20	16	12	10	8	6	5	4	3	2	1
h(r)	1	1.2	1.5	2	2.4	3	4	4.8	6	8	12	24

*

Line 6 from bottom of page 5-152. This principle can be proved in an entirely analogous manner to that in which the principle on page 5-131 is proved. See the COMMENTARY for 5-131.

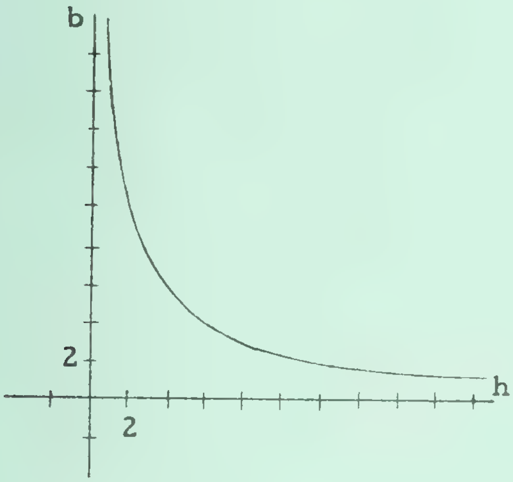
Answers for Part A [which begins on page 5-153].

1. 16 2. 16 3. 15 4. 4 5. 4 6. $\frac{4}{3}$ 7. 28

*

Answers for Part B [on pages 5-154 and 5-155].

1.



2. Yes; $f = \{(x, y): \; y = \frac{24}{x}\}$

3. No

5. If M varies directly as N and $0 \notin \mathcal{R}_N$ then M varies inversely as $1/N$, and the factor of inverse variation is the same as the factor of direct variation. [If $0 \in \mathcal{R}_N$, then $\mathcal{D}_{1/N} \neq \mathcal{D}_N$, so $\mathcal{D}_{1/N} \neq \mathcal{D}_M$. Hence, in this case, M is not inversely proportional to $1/N$ because M and $1/N$ do not have the same domain.]

6. If M varies inversely as N then $\mathcal{D}_M = \mathcal{D}_N$, and there is a constant variable quantity $k \neq 0$ such that $MN = k$. In this case, $0 \notin \mathcal{R}_N$, so $\mathcal{D}_{1/N} = \mathcal{D}_N = \mathcal{D}_M$, and $M = k \cdot \frac{1}{N}$. So, by definition, M varies directly as $1/N$.

*

Answers for Part D [on pages 5-155 and 5-156].

1. Since F varies directly as W, there is a number $k \neq 0$ such that, for each body e, $F(e) = k \cdot W(e)$. For a body e_o such that $W(e_o) = 10$, $F(e_o) = 3$. So, $3 = k \cdot 10$, and $k = 0.3$. Hence, if $W(e) = 25$ then $F(e) = 0.3 \cdot 25 = 7.5$. The frictional force on a 25-lb. block is 7.5 pounds.

[Alternative solution. Since F varies directly as W and $0 \notin \mathcal{R}_W$, for any bodies e_1 and e_2,

$$\frac{F(e_2)}{W(e_2)} = \frac{F(e_1)}{W(e_1)}.$$

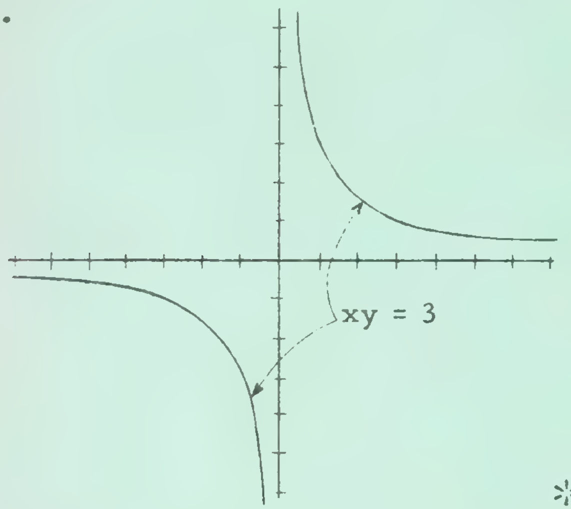
If $W(e_1) = 10$ then $F(e_1) = 3$. So, if $W(e_2) = 25$,

$$\frac{F(e_2)}{25} = \frac{3}{10}.$$

Hence, if $W(e_2) = 25$ then $F(e_2) = 7.5$.]

☆ 4. Three ordered pairs which belong to each function which b is of h are (1, 24), (2, 12), and (3, 8). There is just one linear function which contains the first two of these three points, and its slope is -12. The linear function which contains the second and third points has slope -4. So, no linear function contains all three points, and b is not a linear function of h.

5.



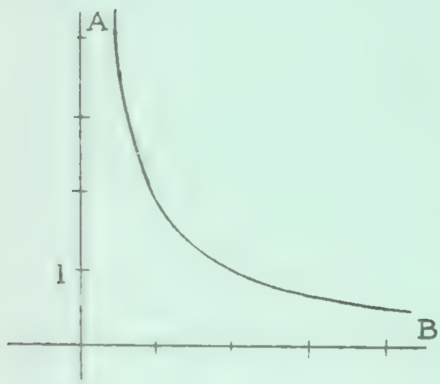
The domain and range of this function are the set of nonzero real numbers.

*

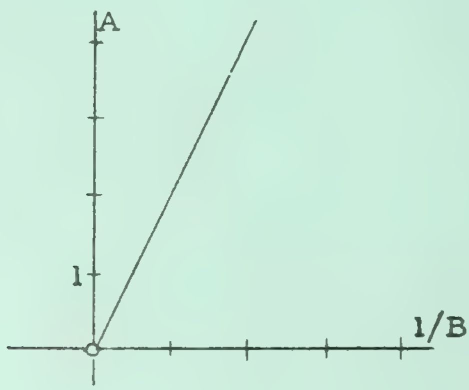
Answers for Part C.

1. $A = \frac{2}{B}$

2.



3.



4. Yes; $\{(x, y): y = 2x\}$

Correction. On page 5-158, in line 12 change 'quantities ℓ and W' to 'quantities ℓ and w'.

2. The factor of direct variation [in Exercise 1] is 0.3.

3. $F = 0.3W$; 30 pounds

*

Answers for Part E.

1. \$6.12 2. \$7.70 3. $S = 0.9L$ 4. 10% [or: $\frac{1}{10}$]

*

Answers for Part F [on pages 5-156 and 5-157].

1. Yes [more or less] 2. 0.0278; $c = 0.0278d$

3. Yes [factor of proportionality $= 1.1 \times 0.0278$]

4. No $\left[\frac{52}{32.6} \neq \frac{27}{14.1} \right]$

*

Answers for Part G [on page 5-157].

1. (a) $3\frac{3}{4}$ hours (b) $H = \frac{30}{M}$ 2. (a) $H = \frac{32}{M}$ (b) 4 hours

*

Answers for Part H [on page 5-157].

1. $\frac{340}{31}$ inches 2. $A = \frac{31}{2}w$ 3. $w = \frac{2}{31}A$ 4. $\frac{31}{2}$; $\frac{2}{31}$

*

Quiz.

1. Suppose that the variable quantity Y is inversely proportional to the variable quantity X. If x_1 and y_1 are corresponding values of X and Y, and $x_1 = 27$ and $y_1 = 3$, what is the factor of inverse proportionality?

2. Grade I apples cost 1.5 times as much as Grade II apples. If you can buy p pounds of Grade I apples for c cents, how many pounds of Grade II apples can you buy for c cents?

3. (a) A varies directly as B and B varies inversely as C. If the factor of direct variation is k_1 and the factor of inverse variation is k_2, write a formula for A in terms of C.

   (b) Does C vary directly as A or inversely as A?

*

Answers for Quiz.

1. 81 2. 3p/2 3. (a) $A = \frac{k_1 k_2}{C}$ (b) inversely

Answers for Part A [which begins on page 5-160].

1. $z = kxy^2$
2. $z = kx^3/y$
3. $z = kxy/w$
4. $V = kr^2h[\pi/3]$
5. $V = kr^3 [k = 4\pi/3]$
6. $F = km_1m/d^2$
7. $I = kE/R$
8. $V = ks^2h$
9. (a) $v = kt$ (b) $s = kt^2$
10. (a) $w = kbd^2/\ell$ (b) $D = kw\ell^3/(bd^3)$

*

Answers for Part B [on pages 5-161 and 5-162].

1. (a) 14 (b) 80
2. (a) 6/5 (b) 100
3. (a) 1/2

   (b) Suppose k_1 and k_2 are nonzero numbers such that $N = k_1R^2$ and $P = k_2R$. Then, since $M = \frac{1}{2} \cdot \frac{N^2}{P^3}$, it follows that

   $$M = \frac{1}{2} \cdot \frac{{k_1}^2R^4}{{k_2}^3R^3} = \frac{{k_1}^2}{2{k_2}^3} \cdot R.$$

   This last result tells us that M varies directly as R.

   (c) Suppose k_3 and k_4 are nonzero numbers such that $N = k_3\sqrt{S}$ and $P = \frac{k_4}{\sqrt[3]{S}}$. Then, since $M = \frac{1}{2} \cdot \frac{N^2}{P^3}$, it follows that

   $$M = \frac{1}{2} \cdot \frac{{k_3}^2S}{\frac{{k_4}^3}{S}} = \frac{{k_3}^2}{2{k_4}^3} \cdot S^2.$$

   This last result tells us that M varies directly as the square of S.

4. (a) 3375 pounds (b) $\frac{2205\pi}{64}$ pounds

☆5. about 687

*

[We suggest that students do Part O of the Miscellaneous Exercises on pages 5-229 and 5-230 as preparation for the exercises on pages 5-163, 5-164, and 5-165.]

*

Quiz.

1. Suppose that A varies directly as B and inversely as the square of C.

   (a) Write a formula which expresses A in terms of B and C. [Use 'k' to name the constant whose value is the factor of variation.]

   (b) If $A(e_1) = 3$, $B(e_1) = 6$, and $C(e_1) = 2$ then $k =$ __?__.

   (c) Find $B(e_2)$ if $A(e_2) = 7$ and $C(e_2) = 4$.

2. What change takes place in the perimeter of an equilateral triangle if you increase the side-measure by 50%?

3. The height-measure of a circular cone varies directly as the volume-measure and inversely as the square of the radius of the base. If you increase the volume-measure by 300% but leave the height-measure unchanged, what change takes place in the radius of the base?

*

Answers for Quiz.

1. (a) $A = \frac{kB}{C^2}$ (b) 2 (c) 56

2. increases by 50%

3. increases by 100%

Answers for Part C.

1. triples
2. decreases by 50%
3. increases by 80%
☆ 4. increases by x%

*

Answers for Part D [on page 5-164].

1. quadruples
2. increases by 96%
3. decreases by 51%
☆ 4. increases by $(2x + \frac{x^2}{100})\%$

*

Answers for Part E [on page 5-164].

1. quadruples
2. decreases by 25%
3. nothing
4. nothing
5. increases by 60%
6. decreases by 37.5%

☆ 7. increases by $(x + y + \frac{xy}{100})\%$

*

Answers for Part F [on pages 5-164 and 5-165].

1. $V = kr^2h$
2. doubles
3. quadruples
4. decreases by $77\frac{7}{9}\%$
☆ 5. increases by $(2x + \frac{x^2}{100})\%$

*

Answers for Part G [on page 5-165].

1. $\ell_2 = \ell_1/3$
2. halved
3. increases by 237.5%
4. increases by 12.5%
5. decrease it by $33\frac{1}{3}\%$

☆ 6. increases by $(p + q + r + \frac{pq + qr + rp}{100} + \frac{pqr}{10000})\%$

*

Answers for Part H [on page 5-165].

1. decreases by 75%
2. none

*

Quiz.

1. Each of the relations described below is a quadratic function. Give the defining equation for each.

   (a) $\{(x, y): y = 6 - 2x + 3x^2\}$; $y =$ ___x^2 + ___x + ___

   (b) $\{(x, y): y = (x - 3)^2\}$; $y =$ ___x^2 + ___x + ___

   (c) $\{(x, y): y = 5 - (2x + 1)^2\}$; $y =$ ___x^2 + ___x + ___

   (d) the function which maps each real number on 3 more than its square; $y =$ ___x^2 + ___x + ___

   (e) $\{(x, y): y = (50 - 2x)x\}$; $y =$ ___x^2 + ___x + ___

2. Find the ordered pair whose graph is the lowest point in the graph of $\{(x, y): y = (x - 2)^2 + 5\}$.

*

Answers for Quiz.

1. (a) 3, −2, 6 (b) 1, −6, 9 (c) −4, −4, 4

   (d) 1, 0, 3 (e) −2, 50, 0

2. (2, 5)

Answers for Part A.

1. linear   2. quadratic   3. quadratic   4. none   5. quadratic

6. Neither constant, linear, nor quadratic; but, a subset of a quadratic function.

7. quadratic   8. quadratic   9. constant $[\{(x, y): y = 14\}]$

*

[In the answers which follow, the parabolas are all congruent to the graph of $\{(x, y): y = x^2\}$. Of course, this assumes that the graphs are drawn with respect to axes which have the same scale for all problems. We identify the parabola just by giving the extreme point and telling whether the parabola opens upward or downward.]

*

Answers for Part B.

1. f [(0, 0), up];   g [45° -line];

   h [parallel to and 1 unit above the horizontal axis]

2. (a) (0, 1), up   (b) (0, 2), up   (c) (0, −5), up

   (d) $(-\frac{1}{2}, -\frac{1}{4})$, up   (e) (−1, −1), up   (f) (3, −9), up

   (g) (0, 0), down   (h) (0, −2), down   (i) (−2, 4), down

*

Answers for Part C.

1. (0, 0), up   2. (3, 0), up   3. (−1, 0), up

4. (0, 1), up   5. (3, 1), up   6. (−1, −5), up

Answers for Part D. [Use '$\forall_{x \neq 0}\ x^2 > 0$' to justify answers.]

1. 0   2. 3   3. −1   4. 0   5. 3   6. −1

*

Answers for Part E.

1. (3, 5), up   2. (5, −9), up   3. (0, 0), down

4. (3, 0), down   5. (−7, 4), down   6. (1, 0), up

*

Answers for questions on page 5-171.

The intersection of a quadratic function and the y-axis consists of a single point, (0, c).

Knowing the axis of symmetry is useful because, knowing this, we can find the extreme point, and also halve the labor of graphing the function.

$\{(x, y): y = 9\} \cap \{(x, y): y = x^2\} = \{(3, 9), (-3, 9)\}$

The length of the interval $\overline{(3, 9)(-3, 9)}$ is 6, and its midpoint is (0, 9).

The set of all such midpoints is $\{(x, y): x = 0 \text{ and } y \geq 0\}$.

Consider some pairs of arguments of the squaring functions which have a constant difference, say, 0 and $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{3}{4}$, $\frac{1}{2}$ and 1, $\frac{3}{4}$ and $\frac{5}{4}$, and 1 and $\frac{3}{2}$. For each pair, compute the difference of the values of the squaring function for the two arguments.

$$(\tfrac{1}{2})^2 - 0^2 = \tfrac{1}{4}, \quad (\tfrac{3}{4})^2 - (\tfrac{1}{4})^2 = \tfrac{1}{2}, \quad 1^2 - (\tfrac{1}{2})^2 = \tfrac{3}{4}$$

$$(\tfrac{5}{4})^2 - (\tfrac{3}{4})^2 = 1, \quad (\tfrac{3}{2})^2 - 1^2 = \tfrac{5}{4}$$

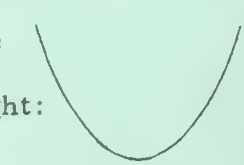
Note that these differences increase. The larger the first argument is, the larger is the difference. This is so because, for each x,

$$(x + \tfrac{1}{2})^2 - x^2 = x + \tfrac{1}{4}.$$

And, for each difference h, for each x, $(x + h)^2 - x^2 = 2hx + h^2$. So, [if $h > 0$,] the larger the first argument, the larger the difference. [Another way of seeing this is by noting that, for each x_1 and x_2, ${x_2}^2 - {x_1}^2 = (x_2 + x_1)(x_2 - x_1)$.]

*

It will pay to draw, on the board and using a large scale, a graph of the squaring function. By graphing such ordered pairs as (1/2, 1/4), (1/4, 1/16), and (−1/4, 1/16), bring out the "flatness" of the function at its extreme point. Graphs of quadratic functions should look like the one on the right:



and not like the one below:

The purpose of the discussion on pages 5-172, 5-173, and 5-174 is to bring out the fact that the graphs of all quadratic functions for which $a = 1$ or $a = -1$ are congruent. They can all be drawn [to the same scale] by using a single parabolic ruler.

*

Answers for questions on page 5-172.

Function	Axis of symmetry	Extreme point
$\{(x, y): y = x^2 - 5\}$	the y-axis	$(0, -5)$
$\{(x, y): y = -x^2\}$	the y-axis	$(0, 0)$
$\{(x, y): y = -x^2 + 5\}$	the y-axis	$(0, 5)$
$\{(x, y): y = -x^2 - 4\}$	the y-axis	$(0, -4)$

For $\{(x, y): y = -(x^2 + 4)\}$, the questions have already been answered, since $\{(x, y): y = -(x^2 + 4)\} = \{(x, y): y = -x^2 - 4\}$.

*

Fill-ins for the six blanks at the bottom of page 5-172.

$x = 0$; $(0, q)$; upward; $x = 0$; $(0, q)$; downward

*

Answers for questions on page 5-173.

Function	Axis of symmetry	Extreme point
$\{(x, y): y = (x - 2)^2\}$	$\{(x, y): x = 2\}$	$(2, 0)$
$\{(x, y): y = -(x - 2)^2\}$	$\{(x, y): x = 2\}$	$(2, 0)$
$\{(x, y): y = (x + 3)^2\}$	$\{(x, y): x = -3\}$	$(-3, 0)$
$\{(x, y): y = -(x + 3)^2\}$	$\{(x, y): x = -3\}$	$(-3, 0)$
$\{(x, y): y = (x - 2)^2 + 5\}$	$\{(x, y): x = 2\}$	$(2, 5)$

*

Fill-ins for the six blanks at the bottom of page 5-173.

$x = p$; $(p, 0)$; upward; $x = p$; $(p, 0)$; downward

Fill-ins for the six blanks at the top of page 5-174.

$x = p$; (p, q); upward; $x = p$; (p, q); downward

*

A graph of $\{(x, y): y = (x - 3)^2 + 5\}$ can be obtained by shifting a graph of the squaring function 3 units to the right and 5 units up.

*

Answers for questions on page 5-174.

The axis of $\{(x, y): y = x^2 - 6x\}$ is $\{(x, y): x = 3\}$. Its extreme point is $(3, -9)$.

[Fill-in: $x = p$]

(a) $\{(x, y): x = 2\}$; $(2, -4)$

(b) $\{(x, y): x = -2\}$; $(-2, -4)$

(c) [same as for (a)]

(d) [same as for (b)]

(e) $\{(x, y): x = \frac{5}{2}\}$; $(\frac{5}{2}, -\frac{25}{4})$

(f) $\{(x, y): x = -\frac{5}{2}\}$; $(-\frac{5}{2}, -\frac{25}{4})$

(g) $\{(x, y): x = \frac{5}{2}\}$; $(\frac{5}{2}, -\frac{13}{4})$

(h) $\{(x, y): x = -\frac{5}{2}\}$; $(-\frac{5}{2}, -\frac{53}{4})$

(i) $\{(x, y): x = \frac{3}{2}\}$; $(\frac{3}{2}, \frac{9}{4})$

(j) $\{(x, y): x = -\frac{3}{2}\}$; $(-\frac{3}{2}, \frac{9}{4})$

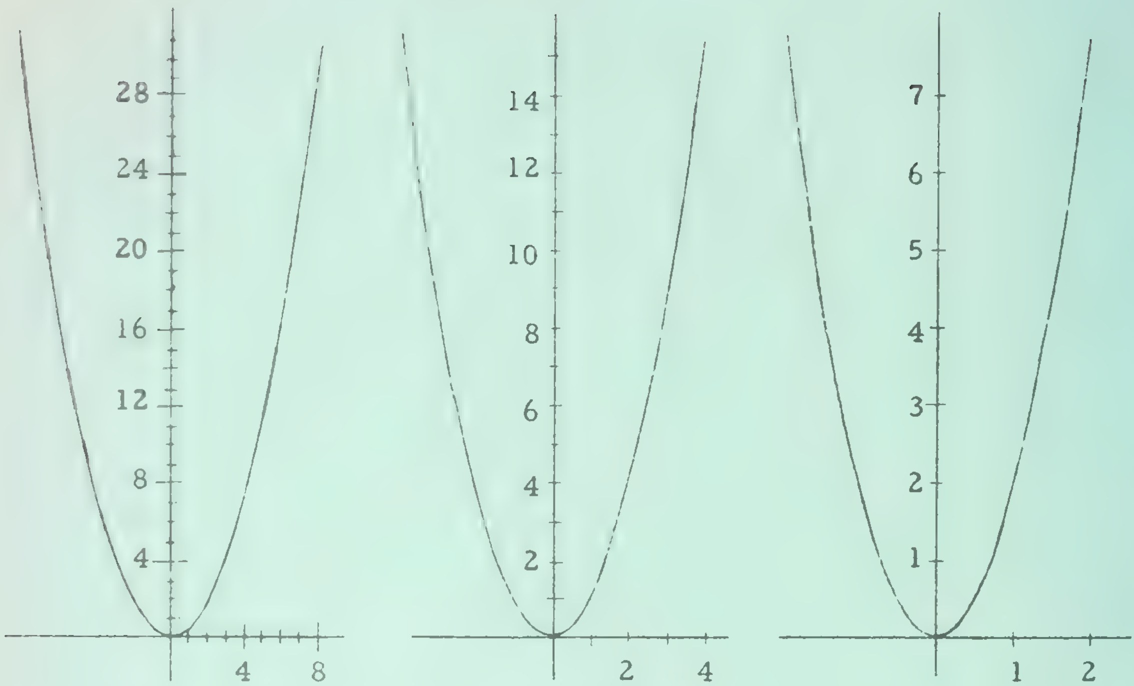
(k) $\{(x, y): x = \frac{3}{2}\}$; $(\frac{3}{2}, \frac{25}{4})$

(ℓ) $\{(x, y): x = -\frac{3}{2}\}$; $(-\frac{3}{2}, -\frac{11}{4})$

*

Fill-ins for the five blanks at the bottom of page 5-174.

$x = -\frac{b}{2}$, $x = -\frac{b}{2}$; $(-\frac{b}{2}, -\frac{b^2}{4} + c)$; upward; $(-\frac{b}{2}, -\frac{b^2}{4} + c)$



$y = \frac{1}{2}x^2$ $\qquad$ $y = x^2$ $\qquad$ $y = 2x^2$

the graph of (u, v) becomes the graph of $(\frac{u}{a}, \frac{v}{a})$, one has a graph of

$\{(x, y): y = ax^2 + bx + c\}$. In case $a > 0$, the required scale is, of course, obtained merely by taking a unit a times as long as the old.

[If $a < 0$, changing the graph of (u, v) into the graph of $(\frac{u}{a}, \frac{v}{a})$ involves

reversing the directions of the axes, as well as using a unit $|a|$ times the length of the original unit. This amounts to a reflection through the origin, as well as a change of unit. So, to get the graph of

$$\{(x, y): y = ax^2 + bx + c\}$$

when $a < 0$, one must reflect the graph of $\{(x, y): y = x^2 + bx + ac\}$ through the origin as well as change to a new unit which is $|a|$ times as long as the old.]

Answers for questions on page 5-177.

(a) vertex at $(2, -1)$, pointing down (b) vertex at $(2, 7)$, pointing up

(c) vertex at $(-\frac{3}{2}, -\frac{1}{2})$, pointing down (d) vertex at $(-\frac{3}{2}, 4)$ pointing up

(e) vertex at $(-3, -11)$, pointing down (f) vertex at $(-3, 7)$, pointing up

(g) vertex at $(-2, 1)$, pointing down (h) vertex at $(-2, 9)$, pointing up

*

Fill-ins for the four blanks at the bottom of page 5-177.

$\{(x, y): x = -\frac{b}{2a}\}$; $(-\frac{b}{2a}, -\frac{b^2 - 4ac}{4a})$; $a > 0$; $a < 0$

*

Students should see that if one uses the same scale on both axes in graphing several quadratic functions, with the same values of 'b' and 'c' in each case, but different values for 'a', then the graphs corresponding with values of 'a' of larger absolute value will be the narrower. In particular, the graphs corresponding with values of 'a' greater than 1 [or less than -1] will be narrower and more pointed than the graph of the squaring function, and those corresponding with values of 'a' between -1 and 1 will be broader and flatter than the graph of the squaring function. However, if one keeps the same scale on the x-axis and, for each graph, uses a different unit, $1/|a|$ times as long, on the y-axis, then the graphs will be the same size and shape as the graph of the squaring function. Some students may make a more fundamental discovery than this. It is that if one graphs the squaring function, using the same unit on both axes, and then graphs any quadratic function $\{(x, y): y = ax^2 + bx + c\}$ using, on <u>both</u> axes, a unit $|a|$ times as long as the original one, then the two graphs will have the same size and shape. In geometrical terms, all parabolas are similar geometric figures. Algebraically, the reason for this is that, for $a \neq 0$,

$$\text{'}y = ax^2 + bx + c\text{' and '}ay = (ax)^2 + b(ax) + ac\text{'}$$

are equivalent equations. Now, $(u, v) \in \{(x, y): y = x^2 + bx + ac\}$ if and only if $(\frac{u}{a}, \frac{v}{a}) \in \{(x, y): ay = (ax)^2 + b(ax) + ac\}$; that is, if and only if $(\frac{u}{a}, \frac{v}{a}) \in \{(x, y): y = ax^2 + bx + c\}$. So, if one graphs $\{(x, y): y = x^2 + bx + ac\}$, using the same scale on both axes, and then changes the scales so that

You may feel that solving quadratic equations by "completing the square" is not a very efficient procedure as compared, say, to the use of the quadratic formula. However, transforming a quadratic expression by completing the square has many important applications, aside from the solution of quadratic equations.

In more advanced courses students will frequently find it necessary to convert an expression such as '$ax^2 + bx + c$' into the simpler form '$au^2 + q$' by completing the square and substituting 'u' for '$x + \frac{b}{2a}$' and 'q' for '$\frac{4ac - b^2}{4a}$'. In fact, they will probably go further and simplify '$2x^2 - 12x + 22$', for example, to '$v^2 + 2^2$', where $v = \sqrt{2}(x - 3)$. It will be important that they know that each quadratic expression can be transformed, in this way, into one of the four forms:

$$v^2 + d^2, \qquad -(v^2 + d^2), \qquad v^2 - d^2, \qquad d^2 - v^2,$$

and that they be skilled in carrying out such transformations. So, it is essential that students now become adept at completing the square.

*

Fill-ins for the nine blanks on page 5-178.

$x - p$; q; $\{(x, y) : x = p\}$; (p, q); $a > 0$; q; $a < 0$; > 1; $|a| < 1$

Correction. In line 12, delete the colon after 'expression'.

Answers for the five 'Why?'s on page 5-180.

dpma; ℓdpma; cpm and apa; dpma; $3 + 3 = 6$

*

Answers for Part A [on pages 5-180 and 5-181].

1. $x^2 - 4x + 4$	2. $y^2 + 8y + 16$	3. $z^2 - 16z + 64$
4. $x^2 - 2x + 1$	5. $p^2 - 14p + 49$	6. $y^2 + 18y + 81$
7. $3x^2 + 30x + 75$	8. $-5x^2 + 40x - 80$	9. $-x^2 + 8x - 16$

[The answer for Exercise 10 is given in the text.]

11. $x^2 - x + \frac{1}{4}$	12. $x^2 + 3x + \frac{9}{4}$	13. $y^2 + 5y + \frac{25}{4}$
14. $z^2 - 7z + \frac{49}{4}$	15. $x^2 - \frac{x}{2} + \frac{1}{16}$	16. $x^2 + \frac{5x}{2} + \frac{25}{16}$
17. $x^2 - kx + \frac{k^2}{4}$	18. $y^2 + my + \frac{m^2}{4}$	19. $x^2 - \frac{2mx}{n} + \frac{m^2}{n^2}$
20. $x^2 - \frac{6x}{m} + \frac{9}{m^2}$	21. $y^2 - \frac{5y}{n} + \frac{25}{4n^2}$	22. $x^2 + \frac{bx}{a} + \frac{b^2}{4a^2}$

[Note that we have omitted restrictions against dividing by 0 in Exercises 19 - 22. Just make a brief mention of this restriction.]

*

In line 5 on page 5-181, we say "most of those in Part A" because the expressions obtained by expanding those in Exercises 8 and 9 of Part A are not perfect squares. [The expression in Exercise 7 is equivalent to '$(\sqrt{3}x + 5\sqrt{3})^2$'.]

*

Answers for Part B [on page 5-181].

1. Yes, $(x+3)^2$	2. No	3. Yes, $(x-2)^2$	4. No
5. Yes, $(x-10)^2$	6. Yes, $(x-6)^2$	7. Yes, $(x+\frac{5}{2})^2$	8. No
9. Yes, $(x-\frac{3}{2})^2$	10. No	11. Yes, $(x+a)^2$	12. No
13. Yes, $(x-4c)^2$	14. Yes, $(x+\frac{9m}{2})^2$		15. No
16. Yes, $(x + \frac{b}{2a})^2$			

Answers for Part C [on pages 5-181 and 5-182].

1. $x^2 + 8x + 4^2$; $\forall_x\ (x + 4)^2 = x^2 + 8x + 4^2$

2. $x^2 + 2x + 1^2$; $\forall_x\ (x + 1)^2 = x^2 + 2x + 1^2$

3. $x^2 - 6x + 3^2$; $\forall_x\ (x - 3)^2 = x^2 - 6x + 3^2$

4. $x^2 - 12x + 6^2$; $\forall_x\ (x - 6)^2 = x^2 - 12x + 6^2$

5. $x^2 + 20x + 10^2$; $\forall_x\ (x + 10)^2 = x^2 + 20x + 10^2$

6. $x^2 + 1000x + 500^2$; $\forall_x\ (x + 500)^2 = x^2 + 1000x + (500)^2$

7. $x^2 - 500x + 250^2$; $\forall_x\ (x - 250)^2 = x^2 - 500x + (250)^2$

8. $x^2 + 4bx + (2b)^2$; $\forall_b \forall_x\ (x + 2b)^2 = x^2 + 4bx + (2b)^2$

9. $x^2 - 6kx + (3k)^2$; $\forall_k \forall_x\ (x - 3k)^2 = x^2 - 6kx + (3k)^2$

10. $x^2 + 3x + (\frac{3}{2})^2$; $\forall_x\ (x + \frac{3}{2})^2 = x^2 + 3x + (\frac{3}{2})^2$

11. $x^2 + 9x + (\frac{9}{2})^2$; $\forall_x\ (x + \frac{9}{2})^2 = x^2 + 9x + (\frac{9}{2})^2$

12. $x^2 - 7x + (\frac{7}{2})^2$; $\forall_x\ (x - \frac{7}{2})^2 = x^2 - 7x + (\frac{7}{2})^2$

13. $x^2 - \frac{x}{2} + (\frac{1}{4})^2$; $\forall_x\ (x - \frac{1}{4})^2 = x^2 - \frac{x}{2} + (\frac{1}{4})^2$

14. $x^2 - kx + (\frac{k}{2})^2$; $\forall_k \forall_x\ (x - \frac{k}{2})^2 = x^2 - kx + (\frac{k}{2})^2$

15. $x - \frac{x}{k} + (\frac{1}{2k})^2$; $\forall_{k \neq 0} \forall_x\ (x - \frac{1}{2k})^2 = x^2 - \frac{x}{k} + (\frac{1}{2k})^2$

16. $x^2 - \frac{4x}{k} + (\frac{2}{k})^2$; $\forall_{k \neq 0} \forall_x\ (x - \frac{2}{k})^2 = x^2 - \frac{4x}{k} + (\frac{2}{k})^2$

17. $x^2 - \frac{3x}{k} + (\frac{3}{2k})^2$; $\forall_{k \neq 0} \forall_x\ (x - \frac{3}{2k})^2 = x^2 - \frac{3x}{k} + (\frac{3}{2k})^2$

18. $x^2 + \frac{bx}{a} + (\frac{b}{2a})^2$; $\forall_{a \neq 0} \forall_b \forall_x\ (x + \frac{b}{2a})^2 = x^2 + \frac{bx}{a} + (\frac{b}{2a})^2$

Answers for Part D [on pages 5-182 and 5-183].

1. $(x+3)^2 - 7$

2. $(x+6)^2 - 39$

3. $(x-4)^2 - 7$

4. $(x-8)^2 - 34$

5. $(x+5)^2 - 14$

6. $(x-2)^2 - 2$

7. $(x+\frac{1}{2})^2 - \frac{29}{4}$

8. $(x-\frac{1}{2})^2 - \frac{1}{4}$

9. $(x+\frac{5}{2})^2 - \frac{9}{4}$

10. $(x-\frac{7}{2})^2 - \frac{57}{4}$

11. $(x+\frac{1}{4})^2 + \frac{79}{16}$

12. $(x-\frac{1}{6})^2 + \frac{35}{36}$

13. $(x+m)^2 + (n-m^2)$

14. $(x-2b)^2 + (d-4b^2)$

15. $(x-\frac{1}{2k})^2 + \frac{4ck-1}{4k^2}$

16. $(x+\frac{b}{2a})^2 + \frac{4ac-b^2}{4a^2}$

17. $2(x-5)^2 - 43$

18. $5(x+2)^2 - 24$

19. $3(x+2)^2 - 13$

20. $2(x+\frac{1}{4})^2 - \frac{73}{8}$

21. $2(x-\frac{3}{4})^2 + \frac{23}{8}$

22. $-5(x-\frac{1}{2})^2 - \frac{11}{4}$

23. $a(x+\frac{5}{2})^2 + \frac{28-25a}{4}$

24. $a(x+\frac{b}{2a})^2 + \frac{4ac-b^2}{4a}$

Quiz.

1. Transform each of the following quadratic expressions in 'x' by completing the square and simplifying.

Sample. $x^2 + 7x + 2$

Solution. $(x + \frac{7}{2})^2 - \frac{41}{4}$

(a) $x^2 + 10x + 1$ (b) $x^2 - 18x - 3$

(c) $x^2 + x + 3$ (d) $-x^2 + 5x + 7$

(e) $x^2 + tx$ (f) $2x^2 - 12x + 5$

2. (a) Find the ordered pair of the function $\{(x, y): y = x^2 + 9x - 8\}$ whose second component is the extreme value of the function.

(b) Write an equation of the axis of symmetry of the function $\{(x, y): y = 8 + x(5 - x)\}$.

*

Answers for Quiz.

1. (a) $(x + 5)^2 - 24$ (b) $(x - 9)^2 - 84$ (c) $(x + \frac{1}{2})^2 + \frac{11}{4}$

(d) $-(x - \frac{5}{2})^2 + \frac{53}{4}$ (e) $(x + \frac{t}{2})^2 - \frac{t^2}{4}$ (f) $2(x - 3)^2 - 13$

2. (a) $(-\frac{9}{2}, -\frac{113}{4})$ (b) $x = \frac{5}{2}$

Answers for Part E.

1. $(-\frac{9}{2}, -\frac{113}{4})$; min

2. $(-\frac{9}{2}, \frac{113}{4})$; max

3. $(\frac{1}{5}, \frac{34}{5})$; min

4. $(\frac{5}{14}, \frac{277}{28})$; max

5. $(-\frac{1}{6}, \frac{11}{12})$; min

6. (1, 0); min

7. (2, 13); min

8. [There is no extreme point.]

9. There are two extreme points, $(-1/2, 27/4)$ and (2, 13). The minimum value of the function is 27/4; the maximum value of this function is 13. [Exercises 7, 8, and 9 of Part E deal with functions which, while not themselves quadratic functions, are subsets of the quadratic function $\{(x, y) : y = x^2 + x + 7\}$. Graphing this quadratic function will help in explaining the answers for these three exercises. For Exercise 7, note that (2, 13) belongs to the function in question and that the graph of this point is the lowest point on the graph of the function. The range of the function is $\{y : y \geq 13\}$. For Exercise 8, note these three things:

   (1) (2, 13) does not belong to the graph of the function in question,

   (2) although there are values of the function as close to 13 as one wishes, all values of the function are greater than 13,

   (3) the function has no least value.

   The range of the function in Exercise 8 is $\{y : y > 13\}$, and the range of the function in Exercise 9 is $\{y : 27/4 \leq y \leq 13\}$.]

*

Answers for Part F [on pages 5-183 and 5-184].

1. $\forall_x (120 - 2x)x = -2(x - 30)^2 + 1800$. So, the extreme point of the function is (30, 1800). Hence, the pen should be 30 feet wide [and 60 feet long].

2. 25 yards by 50 yards

*

Corrections. [for page 5-185]
line 1: change 'a gardener' to 'A gardener'
line 6b: insert 'than' between 'sytematic' and 'the'

3. 100 yards by 150 yards

4. $2\frac{1}{2}$ inches

5. 63 [If one plots total yield against number of trees, one obtains points of the graph of $\{(x, y) : y = x[400 - (x - 60)6]\}$. Since the maximum of this quadratic function corresponds with the argument $63\frac{1}{3}$ [and since $63\frac{1}{3} \geq 60$], 63 trees will produce the largest crop.]

6. In terms of its width-measure x, the length-measure of a rectangle of perimeter p is $\frac{p}{2} - x$, and its area-measure is $(\frac{p}{2} - x)x$. The argument corresponding with the maximum of $\{(x, y): y = \frac{px}{2} - x^2\}$ is $\frac{p}{4}$. So, the rectangles of perimeter p which have the greatest area are those whose width-measure is $\frac{p}{4}$--that is, those which are squares.

7. Since $\overleftrightarrow{AP}$ and $\overleftrightarrow{AC}$ have the same slope [or: by similar triangles], if x is the width-measure of the foundation and y is its length-measure, $\frac{y}{40 - x} = \frac{60}{40}$. So, the width-measure of the desired foundation is the argument of the extreme point of $\{(x, A) : A = \frac{3}{2}(40 - x)x\}$. This argument is 20. So, the corner on $\overleftrightarrow{AB}$ should be 20 feet from B and the corner on $\overleftrightarrow{BC}$ should be 30 feet from B.

8. 2 weeks after July 1 [The argument of the extreme point of $\{(x, y) : y = (6 + 3x)(2 - \frac{1}{3}x)\}$ is 2.]

*

We suggest that students work on Parts L, M, and N of the Miscellaneous Exercises [pages 5-225ff.] before they do the work on quadratic equations in section 5.10.

In order to prove:

$$(*) \qquad \forall_{x \geq 0} \forall_{y > 0} \sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}},$$

one makes use of the fact that, by definition,

$$\forall_{u \geq 0} \sqrt{u} = \underline{\text{the}}\ v \text{ such that } v \geq 0 \text{ and } v^2 = u.$$

[The logical admissibility of the definition of '$\sqrt{\ }$' is reviewed on TC[5-80] in the discussion for Sample 2.]

Proof of (*):

Since $x \geq 0$ and $y > 0$, it follows that $x \geq 0$ and $y \geq 0$.
So, by definition, since $x \geq 0$, and $y \geq 0$,

$$\sqrt{x} = \text{the } u \text{ such that } u \geq 0 \text{ and } u^2 = x$$

and $$\sqrt{y} = \text{the } v \text{ such that } v \geq 0 \text{ and } v^2 = y.$$

Since $y > 0$, $y \neq 0$. Since $y \neq 0$ and $0^2 = 0$, $\sqrt{y} \neq 0$.
So, $\sqrt{x} \geq 0$ and $\sqrt{y} > 0$. Hence,

$$\frac{\sqrt{x}}{\sqrt{y}} \geq 0.$$

Also $$\left(\frac{\sqrt{x}}{\sqrt{y}}\right)^2 = \frac{(\sqrt{x})^2}{(\sqrt{y})^2} = \frac{x}{y}.$$

Consequently,

$$\frac{\sqrt{x}}{\sqrt{y}} = \text{the } w \text{ such that } w \geq 0 \text{ and } w^2 = \frac{x}{y}.$$

So, by definition, $\dfrac{\sqrt{x}}{\sqrt{y}} = \sqrt{\dfrac{x}{y}}$.

[For additional comments on similar topics, see TC[3-110, 111]a, b, and c.]

Quiz.

1. Solve these equations by completing the square. Show all of your work.

   (a) $x^2 + 4x - 1 = 20$ (b) $x^2 - 5x - 1 = 0$

2. Find the points in the intersection of $\{(x, y): y = 3(x - 1)^2\}$ and the x-axis.

*

Answers for Quiz.

1. (a) $3, \ -7$ (b) $\frac{5 + \sqrt{29}}{2}, \ \frac{5 - \sqrt{29}}{2}$

2. $(1, 0)$

Answers for Part A.

1. $-3 + 4\sqrt{2}$, $-3 - 4\sqrt{2}$

2. $-4 + \sqrt{21}$, $-4 - \sqrt{21}$

3. $\dfrac{-5 + \sqrt{29}}{2}$, $\dfrac{-5 - \sqrt{29}}{2}$

4. $\dfrac{9 + \sqrt{65}}{2}$, $\dfrac{9 - \sqrt{65}}{2}$

5. $1 + 2\sqrt{2}$, $1 - 2\sqrt{2}$

6. 3, -4

7. $\dfrac{-9 + \sqrt{145}}{4}$, $\dfrac{-9 - \sqrt{145}}{4}$

8. $\dfrac{3}{2}$, -1

9. $\dfrac{7 + \sqrt{61}}{6}$, $\dfrac{7 - \sqrt{61}}{6}$

10. $4 + 2\sqrt{3}$, $4 - 2\sqrt{3}$

11. $\dfrac{-1 + \sqrt{19}}{3}$, $\dfrac{-1 - \sqrt{19}}{3}$

12. $\dfrac{-1 + \sqrt{61}}{10}$, $\dfrac{-1 - \sqrt{61}}{10}$

13. -4

14. 5

*

Answers for Part B [on page 5-190].

1. $(-7 + 3\sqrt{21}, 0)$, $(-7 - 3\sqrt{21}, 0)$

2. (12, 0), (8, 0)

3. (4, 0)

4. (5, 0), $(-7, 0)$

5. (5, 0), $(-\frac{3}{2}, 0)$

6. (4, 0)

7. (2, 0), $(-2, 0)$

8. none

9. (5, 0), (1, 0)

10. $(-1, 0)$, (2, 0)

☆11. If $p^2 - 4q \geq 0$, the intersection consists of $\left(\dfrac{-p + \sqrt{p^2 - 4q}}{2}, 0\right)$, and $\left(\dfrac{-p - \sqrt{p^2 - 4q}}{2}, 0\right)$. If $p^2 - 4q < 0$, the intersection is $\emptyset$.

☆12. If $b^2 - 4ac \geq 0$, the intersection consists of $\left(\dfrac{-b + \sqrt{b^2 - 4ac}}{2a}, 0\right)$, and $\left(\dfrac{-b - \sqrt{b^2 - 4ac}}{2a}, 0\right)$. If $b^2 - 4ac < 0$, the intersection is $\emptyset$.

*

Correction. On page 5-192, line 4b:
. . . call this the quadratic formula . . .

Note that the discussion on pages 5-191 and 5-192 shows that, under the restriction '$a \neq 0$ and $b^2 - 4ac \geq 0$' the sentences:

$$ax^2 + bx + c = 0$$

and:

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

are equivalent. For example, subject to the restriction '$a \neq 0$', '$ax^2 + bx + c = 0$' is equivalent to '$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$' because of the principle:

$$\forall_u \forall_{v \neq 0} [u = 0 \Longleftrightarrow u/v = 0]$$ [Th. 53 and Th. 54 on TC[2-61]g],

and the sentences in lines 4 and 5 on page 5-192 are equivalent, subject to the restriction '$a \neq 0$ and $b^2 - 4ac \geq 0$' because of the principle:

$$\forall_u \forall_v [uv = 0 \Longleftrightarrow (u = 0 \text{ or } v = 0)]$$ [The cpm, the pm0, and Th. 56 on TC[2-61]g].

Establishing equivalence is important in justifying the use of the quadratic formula in solving quadratic equations. For, if one were to note only that each of the sentences in question is a consequence of the preceding sentence, then all he would be justified in concluding is that if a quadratic equation [for which the restrictions are satisfied] has a root then it is one of the numbers "given" by the quadratic formula. In this case he would have no reason to believe that a number given by the formula was in fact a root until he had checked by substituting in the given equation. But, once one has seen that the equation and the formula are equivalent, there is no need to check roots given by the formula [except for the purpose of discovering errors in arithmetic].

*

The '$|2a|$' [here, of course, an abbreviation for '$^+|2a|$'] can, as is done in the text, be replaced by '$2a$' for the following reason. The restriction '$a \neq 0$' is equivalent to '$a > 0$ or $a < 0$', giving us two cases. If $a > 0$ then $^+|2a| = 2a$; so, the replacement is justified in this case. If $a < 0$ then $^+|2a| = -2a$; so, in this case, '$^+|2a|$' may be replaced by '$-2a$'. If this is done, one obtains [after a little simplification] the same formulas for the roots as in the first case.

In discussing the exercises of Part A it will be wise to ask the class whether an equation such as '$2a^2 - 7a - 5 = 0$' is equivalent to the equation of Exercise 1. Students need to recognize that, even though the quadratic formula was derived from a quadratic equation of the form '$ax^2 + bx + c = 0$', it may be used to solve any quadratic equation in one variable. Often students think that they cannot use the quadratic formula to solve a quadratic equation for which the variable is 'a' [or 'y', or 'n', or ...].

Also, even though the first example on page 5-193 points out that, in order to use the quadratic formula, it is helpful to first transform a given equation into standard form [if it isn't] and then identify the values to be used for 'a', 'b', and 'c' in the formula, these points will probably need further emphasis. In fact, it may be advisable to assign first the task of transforming to standard form each of the given equations which is not in standard form. Then, ask students to name the values to be used for 'a', 'b', and 'c' in each case. After students become skillful in doing these things, the task of using the formula to obtain roots is relatively easy.

*

Answers for Part A.

1. $\frac{7 + \sqrt{89}}{4}, \frac{7 - \sqrt{89}}{4}$

2. $\frac{-1 + \sqrt{85}}{6}, \frac{-1 - \sqrt{85}}{6}$

3. $\frac{-1 + \sqrt{21}}{10}, \frac{-1 - \sqrt{21}}{10}$

4. $\frac{-3 + \sqrt{6}}{2}, \frac{-3 - \sqrt{6}}{2}$

5. $\frac{3 + \sqrt{65}}{8}, \frac{3 - \sqrt{65}}{8}$

6. 1.6, −1

7. [no real roots]

8. 0 [linear equation]

9. 0, −0.4

10. $\frac{\sqrt{55}}{11}, -\frac{\sqrt{55}}{11}$

11. $\frac{-3 + \sqrt{3}}{2}, \frac{-3 - \sqrt{3}}{2}$

12. [no real roots]

13. $\frac{5 + \sqrt{29}}{2}, \frac{5 - \sqrt{29}}{2}$

[In solving the equation of Exercise 13, it may help the students to suggest that, if '$\square$' were substituted for '$(1 - x)$', the equation would become:

$$\square^2 + 3\square - 5 = 0$$

and, by the quadratic formula, this equation is equivalent to the sentence:

$$\square = \frac{-3 + \sqrt{9 + 20}}{2} \quad \text{or } \square = \frac{-3 - \sqrt{9 + 20}}{2}$$

Hence, the given equation is equivalent to:

$$1 - x = \frac{-3 + \sqrt{9 + 20}}{2} \quad \text{or} \quad 1 - x = \frac{-3 - \sqrt{9 + 20}}{2}$$

which can be simplified to: $x = \frac{5 - \sqrt{29}}{2}$ or $x = \frac{5 + \sqrt{29}}{2}$.]

14. $\frac{1 + \sqrt{46}}{15}$, $\frac{1 - \sqrt{46}}{15}$

[This exercise should be handled in a manner similar to that suggested for Exercise 13. If we use '$\square$' for '$(\frac{1}{x})$', the equation becomes:

$$3\square^2 + 2\square - 15 = 0$$

and, by the quadratic formula, is equivalent to:

$$\square = \frac{-2 + \sqrt{4 + 180}}{6} \quad \text{or} \quad \square = \frac{-2 - \sqrt{4 + 180}}{6}$$

Hence, the given equation is equivalent to:

$$\frac{1}{x} = \frac{-2 + \sqrt{184}}{6} \quad \text{or} \quad \frac{1}{x} = \frac{-2 - \sqrt{184}}{6}$$

which can be simplified to '$x = \frac{3}{-1 + \sqrt{46}}$ or $x = \frac{3}{-1 - \sqrt{46}}$'.

You may wish at this time to show students how, by "rationalizing denominators" this last sentence can be transformed to:

$$x = \frac{1 + \sqrt{46}}{15} \quad \text{or} \quad x = \frac{1 - \sqrt{46}}{15},$$

thereby making it easier to find rational approximations. An alternate procedure is to transform the given equation into '$15x^2 - 2x - 3 = 0$', and apply the quadratic formula.]

☆15. $6 - \sqrt{20}$ [only one root]

[In solving the equation of Exercise 15, use a procedure similar to that for Exercises 13 and 14 to show that, by the quadratic formula, this equation is equivalent to:

$$\sqrt{x} = \frac{-2 + \sqrt{4 + 16}}{2} \quad \text{or} \quad \sqrt{x} = \frac{-2 - \sqrt{4 + 16}}{2}$$

that is, to the sentence '$\sqrt{x} = -1 + \sqrt{5}$ or $\sqrt{x} = -1 - \sqrt{5}$'.

Since a principal square root is never negative, and since $-1 - \sqrt{5} < 0$, this last sentence is equivalent to '$\sqrt{x} = -1 + \sqrt{5}$'. Since $-1 + \sqrt{5} \geq 0$, this equation is equivalent to '$x = (-1 + \sqrt{5})^2$'. [In computing an approximation to $(-1 + \sqrt{5})^2$ it is more advantageous to use the form '$6 - \sqrt{20}$', rather than '$6 - 2\sqrt{5}$'. This is the case because the approximation obtained from the table for either $\sqrt{20}$ or $\sqrt{5}$ may be in error by as much as 0.0005, and multiplying the latter approximation by -2 yields an approximation to $\sqrt{20}$ which may be in error by as much as 2×0.0005.].]

☆16. [no real roots]

[The equation in Exercise 16 can be transformed like this:

$$\left.\begin{array}{l} \sqrt{x} - x - 3 = 0 \\ -x + \sqrt{x} - 3 = 0 \end{array}\right\} \text{ps, cpa}$$

$$\left.\begin{array}{l} -x + \sqrt{x} - 3 = 0 \\ x - \sqrt{x} + 3 = 0 \end{array}\right\} \text{MTP } [-1 \neq 0]$$

$$\left.\begin{array}{l} x - \sqrt{x} + 3 = 0 \\ (\sqrt{x})^2 - \sqrt{x} + 3 = 0 \end{array}\right\} \forall_{u \geq 0}\ \sqrt{u} \text{ is the } v \geq 0 \text{ such that } v^2 = u$$

Since, for this "quadratic equation in '$\sqrt{x}$'", $b^2 - 4ac = 1 - 12 < 0$, there are no values of '$\sqrt{x}$' which satisfy the equation. Hence, there are no values of 'x' which satisfy it.]

*

Answers for Part B [on page 5-194].

1. one	2. two	3. two	4. two	5. two
6. two	7. one	8. none	9. two	

*

It is, of course, incorrect [though commonly done] to say that, for example, '$x^2 - 6x + 9 = 0$' has two roots, each of which is 3. Rather than comparing the <u>number</u> of roots of a polynomial equation with its degree, one should consider the <u>sum of the multiplicities</u> of its roots. For example, '$(x - 3)^2(x + 5)(x - 2)^3 = 0$' is an equation of degree 6 which has, not 6, but only 3 roots. However, one of these is a root of multiplicity 2, a second is of multiplicity 1, and the third is of multiplicity 3; and $2 + 1 + 3 = 6$, the degree of the equation. The notion of multiplicity of roots will be discussed in a later unit.

approximation is at most 0.0005/4, or 0.000125. For Exercise 2, the rounding-off error for the larger root is 0; so, this approximation is in error by at most 0.0005/6, or $0.00008\overline{3}$. But, the rounding-off error for the smaller root is $0.0000\overline{3}$; so, the best we can say is that the approximation is in error by at most $0.00008\overline{3} + 0.0000\overline{3}$, or $0.0001\overline{16}$.]

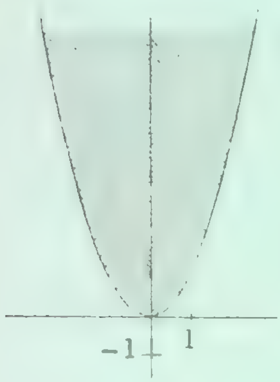
Rational approximations to the roots of certain equations in Part A [on page 5-194].

1. 4.1085, −0.6085 [0.00013]
2. 1.3700, −1.7033 [0.00012]
3. 0.3583, −0.5583 [0.00005]
4. −0.2755, −2.7245 [0.00025]
5. 1.3828, −0.6328 [0.00012]
10. 0.6742, −0.6742 [0.00007]
11. −0.6340, −2.3660 [0.00025]
13. 5.1925, −0.1925 [0.00025]
14. 0.5188, −0.3855 [0.00007]
15. 1.528 [0.0005]

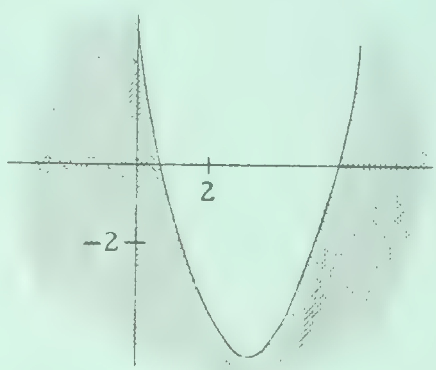
*

Answers for Part ☆F [on page 5-195].

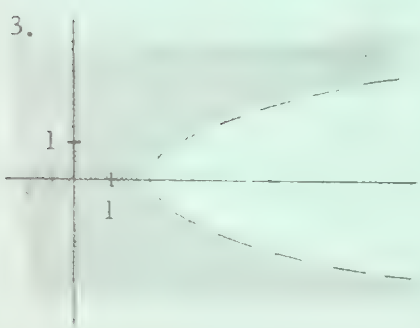
1.



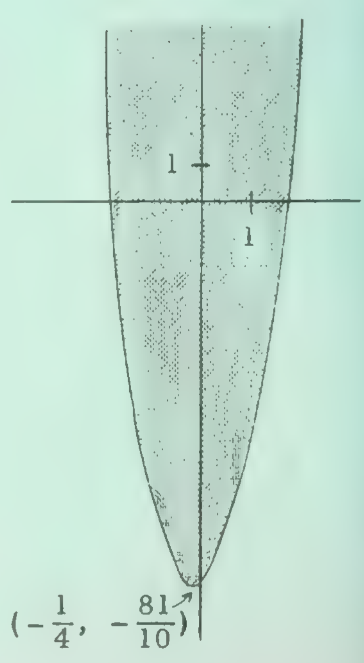
2.



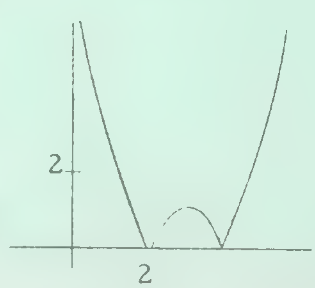
3.



4.



5.



3. 1, −1.7 [1 and $-\frac{5}{3}$ are the exact roots.]

4. 1.3, −0.3 [1.264 and −0.264 are correct to the nearest 0.001. In fact, each is in error by at most 0.00025.]

5. 3.3, −3.3 [3.317 and −3.317 are correct to the nearest 0.001.]

6. 6.5, 1.9 [6.482 and 1.851 are correct to the nearest 0.001. The error in the first is at most $0.00008\overline{3}$; the error in the second is at most $0.00041\overline{6}$.]

*

Answers for Part ☆D [on page 5-195].

1. $x^2 - 7x + 12 = 0$ 2. $x^2 - 10x + 25 = 0$ 3. $x^2 = 0$ 4. $x^2 - 10 = 0$

5. $x^2 - 64 = 0$ 6. $x^2 + \frac{1}{12}x - \frac{1}{2} = 0$, [or: $12x^2 + x - 6 = 0$]

7. $x^2 - (r_1 + r_2)x + r_1 r_2 = 0$ 8. $x^2 - 4x - 1 = 0$

[After solving Exercise 7 of Part D, students should see a short cut for solving Exercise 8, and may find it interesting to use this short cut to check their answers for the earlier exercises.]

*

Answers for Part ☆E [on page 5-195].

1. $\sqrt{2}$, $-2\sqrt{2}$ 2. $\frac{\sqrt{3}+\sqrt{6}}{2}$, $\frac{\sqrt{3}-\sqrt{6}}{2}$

*

If you would like to give your students more practice in finding rational approximations to the roots of quadratic equations whose roots are irrational, you might have them find such approximations for the equations of Exercises 1 - 5, 10, 11, 13, 14, and 15 in Part A on page 5-194. For your convenience, we list approximations to the roots which are correct to the nearest 0.0001. [These approximations were obtained by using the table on page 5-219.] We also give an estimate of the error in these approximations.

[In estimating errors we have used, in each case, estimates of the actual rounding-off error. This is never more than 0.00005, and is usually less. Thus, for Exercise 1, the rounding-off error in the approximations to both roots is 0. So, the total error in each

However, if we round off, and take either 3.608 or 3.609 as an approximation to $\frac{5 + \sqrt{89}}{4}$, then there is an additional error of [in this case] precisely 0.0005. So, the best we can say is that 3.608 [and 3.609] is in error by at most 0.000625.

In general, to estimate the error in a rational approximation to a root obtained by using the quadratic formula, one must add an estimate of the error made in approximating the square root and an estimate of the rounding-off error. If one rounds off the result of the division to 3 decimal places [thus, approximating the quotient correct to the nearest 0.001], the rounding-off error is never more than 0.0005. So, the error in the root is at most $0.0005 + \frac{0.0005}{|2a|}$, or $(1 + \frac{1}{|2a|}) \times 0.0005$. [But, as pointed out in examples which follow, one can usually find a better estimate of the rounding-off error, and so decrease one's estimate of the error in the root.]

If $|a| \geq 1$ then this estimate is at most $\frac{3}{2} \times 0.0005$, or 0.00075. If $a = 2$, the estimate is, as in the example above, $\frac{5}{4} \times 0.0005$, or 0.000625. If one rounds off the division to 4 decimal places [that is, correct to the nearest 0.0001] then the error in the root is at most $0.00005 + \frac{0.0005}{|2a|}$, or $(\frac{1}{10} + \frac{1}{|2a|}) \times 0.0005$. In this case, if $|a| \geq 1$, the estimate is <u>at most</u> $\frac{6}{10} \times 0.0005$, or 0.0003. If $|a| \geq 5$, the estimate is <u>at most</u> 0.0001. These results show that it is worthwhile to round off the division to 4 decimal places [correct to the nearest 0.0001] even when using a 3-place table of square roots. Although the approximations obtained will not in general be correct to the fourth decimal place, they will be more accurate than those obtained by rounding off to 3 decimal places.

*

Answers for Part C [on page 5-195].

1. 6.1, −1.1 [6.14 and −1.14 are correct to the nearest 0.001. In fact, each is in error by at most 0.00025.]

2. 1.3, −6.3 [1.275 and −6.275 are correct to the nearest 0.001. In fact, each is in error by at most 0.00025.]

It may be fruitful to have the class spend some time considering the possible error in the rational approximations to irrational roots of quadratic equations. We know that, for any quadratic equation in one variable, if $b^2 - 4ac \geq 0$, the roots are

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Let's consider the problem of approximating the first of these roots. Suppose y is a rational approximation to $\sqrt{b^2 - 4ac}$ correct to the nearest 0.001. [Such approximations are given in the table on page 5-219.] We know then that

$$-0.0005 < \sqrt{b^2 - 4ac} - y < 0.0005.$$

So,

$$-0.0005 \leq (-b + \sqrt{b^2 - 4ac}) - (-b + y) < 0.0005.$$

Hence,

$$|(-b + \sqrt{b^2 - 4ac}) - (-b + y)| \leq 0.0005,$$

and

$$\left| \frac{-b + \sqrt{b^2 - 4ac}}{2a} - \frac{-b + y}{2a} \right| \leq \frac{0.0005}{|2a|}.$$

Therefore, the absolute error introduced by using an approximation which is correct to the nearest 0.001 is at most $\frac{0.0005}{|2a|}$. [You can easily see that this is also the absolute error introduced by using the same approximation to $\sqrt{b^2 - 4ac}$ in finding a rational approximation to the other root of the equation.] For example, the larger root of the equation of the Sample is $\frac{5 + \sqrt{89}}{4}$. And, as noted in the solution, the table on page 5-219 gives 9.434 as the approximation to $\sqrt{89}$ correct to the nearest 0.001. So, the absolute error in using $\frac{5 + 9.434}{4}$, as an approximation to the root, is $\frac{0.0005}{4}$, or 0.000125. Now,

$$\frac{5 + 9.434}{4} = \frac{14.434}{4}$$

$$= 3.6085.$$

So, 3.6085 differs from the larger root of the equation by at most 0.000125--that is,

$$3.608375 \leq \frac{5 + \sqrt{89}}{4} \leq 3.608625.$$

The numbers 5 and 75 referred to in Sample 1 are, of course, numbers of arithmetic, and the domain of the variable 'x' in equation (*) is the set of numbers of arithmetic. But, as in Unit 3 [See TC[3-55], also TC[5-138, 139]], in solving (*) we pretend that the domain of 'x' is the set of real numbers. Then, the numbers of arithmetic corresponding with such nonnegative roots as are obtained are solutions of (*) in its original interpretation [where the domain of 'x' is the set of numbers of arithmetic].

It need not be the case that all numbers of arithmetic which merely satisfy (*) furnish acceptable solutions of the stated problem. For, since a rectangle cannot have 0 as the measure of either its width or its length, acceptable values of 'x' must also satisfy the restrictions '$x > 0$' and '$x > 5/2$'. So, a complete algebraic statement of the problem consists of two sentences, (*) and '$x > 5/2$'.

*

Remarks somewhat similar to the preceding apply to the Solution for Sample 2. Here, the domain of 'x' is, actually, the set of whole numbers of arithmetic, but, in solving the equation at the top of page 5-197, we pretend that the domain of 'x' is the set of real numbers [See TC[3-64, 65, 66, 67]b]. And, again, in order to get a complete statement of the problem one must adjoin to the equation a restriction: x is a nonzero whole number of arithmetic [or, when we reinterpret the domain of 'x': x is a positive integer].

Quiz.

1. Use the quadratic formula to solve each equation. [Show your work.] If no real number satisfies it, say so.

   (a) $2x^2 - 9x - 6 = 0$ (b) $3t(2 - t) - 6(t + 1) + t = 0$

   (c) $x = 7 + \frac{2}{x}$

2. Find three consecutive positive odd numbers the sum of whose squares is 515.

*

Answers for Quiz.

1. (a) $\frac{9 + \sqrt{129}}{4}$, $\frac{9 - \sqrt{129}}{4}$ (b) no real roots

   (c) $\frac{7 + \sqrt{57}}{2}$, $\frac{7 - \sqrt{57}}{2}$

2. 11, 13, 15

3. If the width of the rectangle is x inches then the length is $19 - x$ inches. So, we look for a root of '$x(19 - x) = 78$' [which, since the width of a rectangle is not 0 inches and is smaller than its length, should also satisfy '$0 < x < 19 - x$', or '$0 < x < 19/2$']. The roots of the equation are 13 and 6. So, the width is 6 inches and the length is 13 inches.

4. If the width is x feet then the length is $500 - x$ feet and the area is $x(500 - x)$ square feet. So, we need to find the first component of the extreme point of the quadratic function $\{(x, y): y = x(500 - x)\}$. This is 250. Since $500 - 250 = 250$, the largest rectangular field whose foot-perimeter is 1000 is a square whose side-length is 250 feet.

5. For some x, the numbers in question are $x - 1/2$ and $x + 1/2$. So, we wish to solve the equation:

$$\frac{1}{x - \frac{1}{2}} + \frac{1}{x + \frac{1}{2}} = \frac{40}{21},$$

or: $20x^2 - 21x - 5 = 0$. The roots are $5/4$ and $-1/5$. Hence the problem has two solutions, $(3/4, 7/4)$ and $(-7/10, 3/10)$.

6. If x is the inch-measure of a side of such a square, then $x^2 + 12 = 4x$, and $x > 0$. But, the quadratic equation has no solution $[(-4)^2 - 4 \cdot 1 \cdot 12 < 0]$. So, there is no such square.

7. If the hiker walked x miles per hour, he walked for $6/x$ hours. $6/(x + 2)$ hours is $1/2$ hour less. So, $6/x - 6/(x + 2) = 1/2$. This equation simplifies to '$x^2 + 2x - 24 = 0$' whose roots are -6 and 4. So, the hiker walked 4 miles per hour for $3/2$ hours.

8. If x is the number named by the numerator, then

$$\frac{x}{x - 2} - \frac{x - 2}{x} = \frac{24}{35}.$$

Reducing to standard form, one obtains '$6x^2 - 47x + 35 = 0$'. Its roots are 7 and 5/6. Hence, the fraction is either '7/5' or '(5/6)/−(7/6)'. [Note that although $(5/6)/-(7/6) = -5/7$, '−5/7' is not a solution. '(5/6)/−(7/6)' ≠ '−5/7'. See TC[1-94, 95]a.]

*

Answers for Part G [which begins on page 5-196].

1. If x is the "middle one" of three consecutive integers, then the integers are $x - 1$, x, and $x + 1$. So, it suffices to find a real integer x such that

$$(x - 1)^2 + x^2 + (x + 1)^2 = 302.$$

This equation simplifies to '$x^2 = 100$', whose roots are 10 and -10. Since both are integers, the problem has two solutions, (9, 10, 11), and (-11, -10, -9). [It is instructive to compare this solution with one beginning: If x is the smallest of three consecutive integers... This leads, eventually, to the equation '$x^2 + 2x - 99 = 0$', which is more difficult to solve than is '$x^2 = 100$'. The moral is: When setting up a problem for algebraic solution, make use of any symmetries which are available.]

2. For each four consecutive odd integers, there is an even integer x such that the four odd integers are $x - 3$, $x - 1$, $x + 1$, and $x + 3$. So, we look for an even integer x such that

$$(x - 3)^2 + (x - 1)^2 + (x + 1)^2 + (x + 3)^2 = 36$$

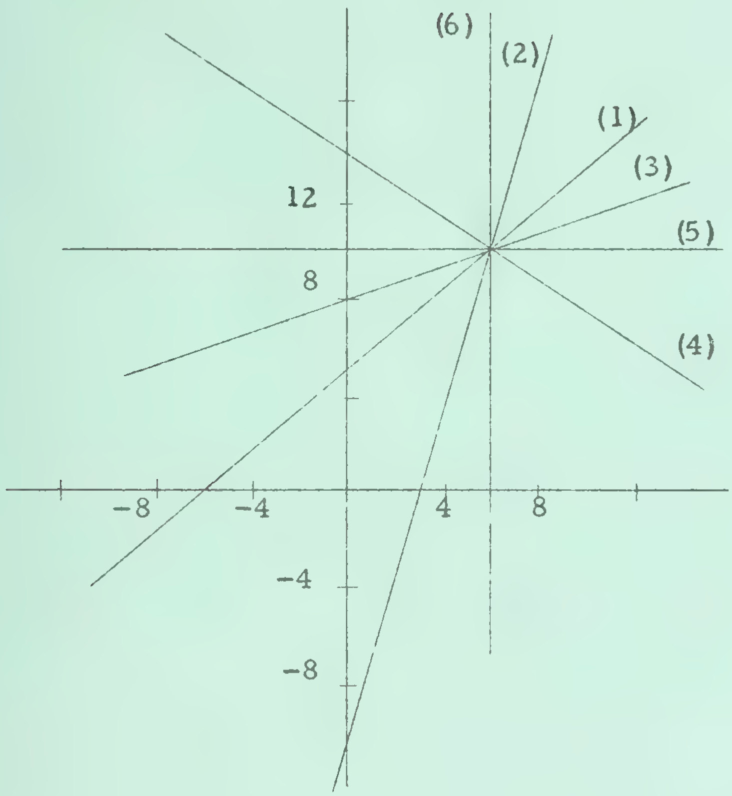
--that is, such that $4x^2 + 20 = 36$. The roots of this equation are 2 and -2, so, since both are even integers, the problem has two solutions, (-1, 1, 3, 5), and (-5, -3, -1, 1).

[Ask if there is a solution for the problem obtained by replacing 'real integers' by 'positive integers'. The answer is, of course, 'no'. Another similar problem, but which has a solution [but just one solution] can be obtained by adding to the exercise 'and the sum of the numbers themselves is positive'. Consideration of such problems should increase students' understanding that a word problem can seldom be completely expressed by one equation. For the given problem, one needs the additional stipulation 'x is an even integer'. For the first suggested modification, one must add 'x is an even integer and $x > 3$'. For the second suggested modification, one needs the additional restriction 'x is a positive even integer'.]

[There are, of course, other ways of setting up the given problem. One may say that there is an integer x such that the consecutive odd integers are $2x - 3$, $2x - 1$, $2x + 1$, and $2x + 3$. Or, one may say that there is an odd integer x such that the four consecutive odd integers are x, $x + 2$, $x + 4$, and $x + 6$ [or: $x - 2$, x, $x + 2$, and $x + 4$]. All these lead to more complicated equations than the one arrived at in the solution given above.]

Answers for Part A.

1., 2., 3.



4. (6, 10); (6, 10); (6, 10)

5. [Students may, correctly, give many answers for each part of this exercise. It is to be hoped, however, that they will see that '(6, 10)' is a correct answer for each part.]

*

Answers for Part B.

1. horizontal
2. vertical
3. horizontal
4. neither
5. horizontal
6. horizontal

*

Answers for Part C.

1. $\boxed{7}$, 5 2. $\boxed{-3}$, 7 3. $\boxed{-3}$, 2

4. $\boxed{5}$, -2 5. $\boxed{1}$, 1 6. $\boxed{1}$, -1

[Other correct answers may be obtained by taking any multiple of the numbers listed. For example, here are three more correct answers for Exercise 2 of Part C:

$$\boxed{3}\ ,\ -7\ ;\ \boxed{-6}\ ,\ 14\ ;\ \boxed{-1}\ ,\ 7/3$$

A similar remark applies to Part D.]

*

Answers for Part D.

1. $\boxed{-2}$, 3 2. $\boxed{5}$, -1 3. $\boxed{8}$, -1

4. $\boxed{4}$, 7 5. $\boxed{9}$, -5 6. $\boxed{9}$, -5

*

Answers for Part E.

1. $(-\frac{3}{31}, \frac{23}{31})$ 2. $(\frac{5}{32}, \frac{13}{32})$ 3. $(12, 5)$

4. $(\frac{31}{43}, -\frac{44}{43})$ 5. $(-\frac{1}{7}, \frac{15}{7})$ 6. $(2, \frac{15}{2})$

Correction. On page 5-202, in line 20, change: '3k - 6m' is 0 to: '-3k - 6m' is 0

Explanation asked for in line 13 on page 5-201.

For each real number x, there is precisely one ordered pair, $(x, -3x + 14)$, whose first component is x and which belongs to the function defined by $(1')$. And, there is precisely one ordered pair, $(x, 5x - 18)$, whose first component is x and which belongs to the function defined by $(2')$. So, a real number x is the first component of an ordered pair which belongs to both functions if and only if

$$(x, -3x + 14) = (x, 5x - 18),$$

that is, if and only if $-3x + 14 = 5x - 18$.

*

Answers for questions at the bottom of page 5-201.

The pm0 tells us that $\forall_k\ k \cdot 0 = 0$.

By the pa0, $0 + 0 = 0$.

*

Answer for question 'Why 0?' in line 7 on page 5-202.

Because this choice will give us an equation equivalent to one which contains only one variable.

*

Notice that the system consisting of equations (3) and (6) is equivalent to that consisting of (3) and (4). [And, so is the system consisting of (4) and (6).] This follows from the easily-proved theorem:

$$\forall_k \forall_{m \neq 0} \forall_p \forall_q\ [(p = 0 \text{ and } q = 0) \text{ if and only if } (p = 0 \text{ and } kp + mq = 0)]$$

So, having found the value of 'y' which satisfies (6), one can use either (3) or (4) to find the corresponding value of 'x'. And, there is no need to check by substituting into the other of these two equations except to catch errors in arithmetic.

Answer for question on page 5-204.

The only need for checking is to catch errors in arithmetic, and checking in (2) may catch errors made in transforming (2) into (2''), as well as any that are made at later stages in the solution.

*

Answers for Part A [on pages 5-204 and 5-205].

[We give, in each case, the solution set.]

1. $\{(-\frac{2}{5}, \frac{1}{5})\}$	2. $\{(5, 4)\}$	3. $\{(9, -9)\}$
4. $\{(0, 0)\}$	5. $\{(\frac{33}{4}, \frac{5}{4})\}$	6. $\{(\frac{17}{7}, \frac{1}{7})\}$
7. $\{(5, 2)\}$	8. $\{(3, -1)\}$	9. $\{(3, -2)\}$
10. $\{(-7, \frac{3}{2})\}$	11. $\{(6, 1)\}$	12. $\{(5, 2)\}$
13. $\{(\frac{22}{3}, \frac{7}{3})\}$	14. $\{(-\frac{7}{4}, -\frac{29}{4})\}$	15. $\{(\frac{41}{39}, \frac{149}{39})\}$
16. $\{(15, 0)\}$	17. $\{(5, 1)\}$	18. $\{(\frac{11}{9}, -\frac{7}{3})\}$
19. $\emptyset$	20. $\{(x, y): 3x + 8y = -2\}$	21. $\{(-\frac{23}{34}, -\frac{39}{17})\}$
22. $\{(\frac{143}{37}, -\frac{58}{37})\}$	23. $\{(-13, 14)\}$	24. $\{(4, 4)\}$

Answers for Part B.

[Reference to $\{(x, y): ax + by + c = 0\}$ and $\{(x, y): a'x + b'y + c' = 0\}$ as linear functions implies that a, b, a', and b' are not 0.]

1. The slopes are $-a/b$ and $-a'/b'$. So, the functions have the same slope if and only if $a/b = a'/b'$--that is [see page 5-145] if and only if a, b, a', b' are in proportion.

2. In this case, there is only <u>one</u> linear function in question.

3. In this case, there are two functions, but they have the same slope.

*

Call to the attention of the students the fact that, in the discussion of independent, dependent, and inconsistent equations, the text deals only with the case of equations which define linear functions. Some of your students may wish to check the fact that the same test for independence [that $ab' \neq a'b$] works for any system of two linear equations in two variables. They can do so by solving Part ☆B on page 5-208.

*

The following diagram may help to sort out the notions of consistency, inconsistency, dependence, and independence.

$$\begin{cases} \text{consistent} \begin{cases} \text{independent} & [ab' \neq a'b] \\ \text{dependent} & \end{cases} \\ \text{inconsistent} \end{cases} \qquad \left.\begin{matrix}\text{dependent} \\ \text{inconsistent}\end{matrix}\right\} [ab' = a'b]$$

Each system of linear equations in two variables is either consistent or inconsistent [but not both]. The equations in a consistent system are either dependent or independent [but not both]. [As is suggested by Exercises 2 and 3 of Part B on page 5-205, one can distinguish between dependent and inconsistent equations [those for which $ab' = a'b$] by evaluating '$bc' - b'c$' and '$ca' - c'a$'. If [when $ab' = a'b$] $bc' = b'c$ and $ca' = c'a$ then the equations are dependent; otherwise, they are inconsistent.]

solved system can be carried out only so far as to yield the system:

$$\left.\begin{aligned}(ab' - a'b)x &= c'b - cb' \\ (ba' - b'a)y &= c'a - ca'\end{aligned}\right\}$$

If $ab' = a'b$ then this system has no solution unless $c'b - cb' = c'a - ca' = 0$. So, if $ab' = a'b$ then the given system has no solution [that is, is inconsistent] unless $c'b = cb'$ and $c'a = ca'$. Finally, if $ab' = a'b$, $bc' = b'c$, and $ca' = c'a$ then the given equations are dependent. [One can see this very easily in case $aba'b' \neq 0$--that is, in case each of the given equations defines a linear function. For, if neither b nor b' is 0, and $ab' = a'b$ and $bc' = b'c$, then $-a'/b' = -a/b$ and $-c'/b' = -c/b$. Hence, in this case, the linear function defined by the second equation has the same slope and intercept as does the linear function defined by the first equation. So, the two equations define the same function. Hence, they have the same solution set--that is, they are dependent.] If $ab' = a'b$, $bc' = b'c$, and $ca' = c'a$ then, for all x and y,

(1) $\qquad a(a'x + b'y + c') = a'(ax + by + c)$

and (2) $\qquad b(a'x + b'y + c') = b'(ax + by + c)$.

If $a \neq 0$ then equation (1) shows that if $ax + by + c = 0$ then $a'x + b'y + c' = 0$, while, if $b \neq 0$, this fact follows from equation (2). Since either $a \neq 0$ or $b \neq 0$, it follows that each solution of the first equation of the given system is a solution of the second equation of the given system. A similar argument, based on the fact that either $a' \neq 0$ or $b' \neq 0$, shows that each solution of the second of the given equations is a solution of the first of them. So, as was to be proved, if $ab' = a'b$, $bc' = b'c$, and $ca' = c'a$ then the given equations have the same solution set--that is, are dependent.

*

Quiz.

Solve these systems of equations.

1. $\left.\begin{aligned}m + n &= -6 \\ m - n &= -10\end{aligned}\right\}$

2. $\left.\begin{aligned}3x - 4y &= 8 \\ x + 2y &= 1\end{aligned}\right\}$

3. $\left.\begin{aligned}6x + 7y &= 3(5 - y) \\ 26 + 10y &= -6x\end{aligned}\right\}$

*

Answers for Quiz.

1. $(-8, 2)$

2. $(2, -\frac{1}{2})$

3. inconsistent

Correction. On page 5-208, in Exercise 15, change '7(x - r)' to '7(s - r)'.
↑

Answers for Part A [which begins on page 5-206].

1. independent $[5 \cdot -7 \neq 6 \cdot 3]$; $(\frac{96}{53}, \frac{52}{53})$

2. independent $[9 \cdot 2 \neq 3 \cdot 7]$; $(\frac{4}{3}, 0)$

3. inconsistent [Since $8 \cdot 6 = -16 \cdot -3$, the equations are either inconsistent or dependent. On transforming the system into:

$$\left.\begin{aligned} s &= \frac{8}{3}t - \frac{7}{3} \\ s &= \frac{8}{3}t - \frac{5}{2} \end{aligned}\right\}$$

it becomes clear that the equations are inconsistent.]

4. independent $[1 \cdot -2 \neq 3 \cdot 1]$; $(0, 0)$

5. independent; $(\frac{1}{3}, \frac{1}{3})$

6. inconsistent

7. inconsistent

8. inconsistent

9. independent; $(5, 0)$

10. dependent

11. inconsistent

12. independent; $(-\frac{53}{10}, -\frac{26}{5})$

13. inconsistent

14. independent; $(3, 0)$

15. independent; $(-\frac{81}{109}, -\frac{36}{109})$

16. dependent

17. independent; $(-\frac{71}{2}, -\frac{3}{8})$

*

Answers for Part ☆B [on page 5-208].

$$\left.\begin{aligned} x &= \frac{c'b - cb'}{ab' - a'b} \\ y &= \frac{c'a - ca'}{ba' - b'a} \end{aligned}\right\}$$

These equations are, of course, not applicable unless $ab' \neq a'b$. If $ab' \neq a'b$ then this "solved" system is equivalent to the given system, and the former, like the latter, has the unique solution

$$\left(\frac{c'b - cb'}{ab' - a'b}, \frac{c'a - ca'}{ba' - b'a}\right).$$

If $ab' = a'b$ then the elimination process which otherwise leads to the

Answers for questions in lower third of page 5-209.

Any solution (x, y, z) of the given system is such that

$$5x + 2y + 3z - 8 = 0$$

$$3x - 4y - z + 14 = 0,$$

and

$$2x + 7y - 3z - 3 = 0.$$

Hence, [by the pm0 and the pa0],

$$[5x + 2y + 3z - 8] + 3[3x - 4y - z + 14] = 0$$

and

$$[5x + 2y + 3z - 8] + [2x + 7y - 3z - 3] = 0.$$

So, if (x, y, z) is any solution of the given system then (x, y) is a solution of the system consisting of (a) and (b).

That (−1, 2, 3) is a solution of the second equation can be seen by substitution. However, such a substitution-check is unnecessary. For, since (−1, 2, 3) is a solution both of equation (a) [considered, now as an equation in 3 variables] and the first equation, it is a solution of:

$$[14x - 10y + 34] - [5x + 2y + 3z - 8] = 0,$$

that is, it is a solution of:

$$9x - 12y - 3z + 42 = 0,$$

which is equivalent to the second equation. Similar remarks, with reference to equation (b) rather than (a), show that (−1, 2, 3) is a solution of the third equation.

We have seen [answer, above, to first question, and because the only solution of the system consisting of (a) and (b) is (−1, 2)] that if (x, y, z) is a solution of the given system then x = −1 and y = 2. Since the only solution (x, y, z) of the first equation for which x = −1 and y = 2 is (−1, 2, 3), no other triple can be a solution of the given system.

*

Answers for Exercises on page 5-210.

1. $\{(1, 2, 3)\}$ 2. $\{(3, 1, 4)\}$ 3. $\{(5, -7, 3)\}$
4. $\emptyset$ [The equations are inconsistent.] 5. $\{(3, -3, \frac{1}{3})\}$
6. $\{(\frac{31}{4}, \frac{7}{4}, -\frac{9}{4})\}$ 7. $\{(-6, -1, -7, 2)\}$ 8. $\{(5, -7, 2, -3)\}$

☆9. $\{(x, y, z): z = 1 \text{ and } x + 2y = -1\}$. [The equations are dependent.]

☆10. $\{(x, y, z): x + y + 4z = 7\}$. [The equations are dependent.]

Correction. On page 5-213, third line from the bottom, change '6w = 202' to '6w = 162'.

Answers for Part A.

1. $\{(\frac{2}{13}, \frac{2}{7})\}$ 2. $\{(\frac{1}{7}, \frac{1}{3})\}$ 3. $\{(\frac{28}{3}, -\frac{5}{3})\}$ 4. $\{(2, 9), (-2, 9)\}$

5. $\{(-4, -\frac{1}{2})\}$ 6. $\{(5, 6), (5, -6), (-5, 6), (-5, -6)\}$

7. $\{(2\sqrt{2}, 2), (2\sqrt{2}, -2), (-2\sqrt{2}, 2), (-2\sqrt{2}, -2)\}$

8. $\{(\sqrt{7}, 3), (-\sqrt{7}, 3)\}$ 9. $\{(-\frac{1}{4}, \frac{1}{30})\}$ 10. $\{(18, 6)\}$

☆11. $\{(1, 2, \frac{1}{3}), (1, -2, \frac{1}{3})\}$ ☆12. $\emptyset$

*

Answers for Part B [on pages 5-212 and 5-213].

1. We must find an equation of the form '$y = ax + b$' which is satisfied by (7, 3) and (−6, 2). In order to do this, we solve the system of equations:

$$\left.\begin{aligned} 3 &= 7a + b \\ 2 &= -6a + b \end{aligned}\right\}$$

The solution set is $\{(\frac{1}{13}, \frac{32}{13})\}$. Hence '$y = \frac{1}{13}x + \frac{32}{13}$' is an equation whose graph is a straight line which contains the graphs of (7, 3) and (−6, 2).

2. $y = 3x^2 + 2x - 5$

☆3. $x = -4y^2 + 3y + 2$

☆4. $\{(4, 29), (2, 3)\}$

*

We suggest assigning Parts Q and S of the Miscellaneous Exercises at this time [5-232 and 5-234ff.] to prepare students for the subsection on problems which starts on page 5-213.

Answers for Exercises [on pages 5-215 through 5-218].

1. 30 dimes, 90 nickels [$d + n = 120$, $10d + 5n = 750$]

2. 67 three-cent stamps, 37 two-cent stamps [r... 3-cent stamps, t... 2-cent stamps; $r + t = 104$, $3r + 2t = 275$]

3. 12 lbs. of 52¢ tea, 8 lbs. of 47¢ tea [x... lbs of 52¢ tea, y... lbs. of 47¢ tea; $x + y = 20$, $52x + 47y = 1000$]

4. The data are inconsistent; the cheapest milk obtainable is 6¢ per quart, so it is impossible to make a mixture costing only 5¢ per quart.

5. 250 cups at 10¢, 600 cups at 5¢ [$f + t = 850$, $5f + 10t = 5500$]

6. 45, 25[$\ell - s = 20$, $5\ell + 7s = 400$]

7. .60['60' is acceptable if one regards '06' as a decimal numeral.] [f... first digit, s... second digit; $f + s = 6$, $0.1(10f + s) = f + 10s$]

8. \$20,000 at 4%, \$5,000 at 3% [x... dollars at 4%, y... dollars at 3%; $x + y = 25000$, $.04x + 03y = 950$]

9. \$3,000 at 6.5%, \$2,000 at 5.5% [n... dollars at 6.5%, d... dollars at 5.5%; $n + d = 5000$, $.065n + .055d = 305$]

10. 72 [$8(t + u) = 10t + u$, $t + 10u = 4t - 1$]

11. \$1,500 at 4%, \$2,000 at 3% [n... dollars at 4%, d... dollars at 3%; $n + 500 = d$, $.04n = .03d$]

12. $7\frac{2}{3}$, $17\frac{1}{3}$ [$x + y = 25$, $y = 2x + 2$]

13. 12, 14 [$\frac{s + \ell}{2} = 13$, $\ell = s + 2$]

14. 80 feet wide, 100 feet long [$2\ell + 2w = 360$, $\ell = w + 20$]

15. 5¢ for cork, \$1.05 for bottle [$c + b = 110$, $b - 100 = c$]

16. (86, −28) [$f + s = 58$, $f - s = 114$]

*

Quiz.

1. Solve this system of equations.

$$\left.\begin{aligned} 3x + y - z &= 10 \\ 2x - 3y - 5z &= -3 \\ 4x + 2y + 7z &= 0 \end{aligned}\right\}$$

2. Three cans of carrots and five cans of peas cost $1.55. Two of these cans of carrots and three cans of the peas cost $.96. What is the price of one can of carrots?

*

Answers for Quiz.

1. (1, 5, −2)

2. 15 cents

Correction. On page 5-218, in the third line of Exercise 29, change 'threr' to 'there'.

17. The data are insufficient. [$3f + 2s = 50$, $.06f + .04s = 1$; since by the MTP the second equation can be transformed into an equation which is a copy of the first, the equations as given by the data are dependent.]

18. 0.5, 5.5[$\ell - s = 5$, $\ell = 3s + 4$]

19. 9 dimes, 28 nickels [$d + n = 37$, $10d + 5n = 230$]

20. $6\frac{1}{4}$, $1\frac{1}{4}$[$\ell - s = 5$, $\ell = 5s$]

21. Andy is $\frac{1}{10}$ year, Bill is $1\frac{3}{5}$ years [$B = 6A + 1$, $2B - 3 = 2A$]

22. Red, 2 for 9¢; green, 2 for 5¢ [$2r + 4g = 19$, $4r + 2g = 23$]

23. \$8,500 at 4.5%, \$14,500 at 3% [n. . . dollars at 4.5%, d. . . dollars at 3%; $n + d = 23000$, $.045n + .03d = 817.50$]

24. 304, 896 [$s + \ell = 1200$, $3s + 2\ell = 2704$]

25. 5 quart jars, 9 pint jars [$q + p = 14$, $2q + p = 19$]

26. The data are inconsistent. [Since the cheaper grade of rice costs 27 cents per pound, it is impossible to make a mixture which costs only 24 cents per pound.]

27. 16 units wide, 24 units long [$2\ell + 2w = 80$, $w = \frac{2}{3}\ell$]

28. The data are inconsistent. [m. . . Mary's age now, j. . . John's age now; $m - 3 = 3(j - 3)$, $m + 3 = 4(j + 3)$. The solution of this system is $(-15, -51)$. But, we are seeking numbers of arithmetic; and, since there is no pair of positive numbers which is a solution of the system, there is no pair of numbers of arithmetic which satisfies the conditions of the problem.]

29. 12 five-dollar bills, 15 one-dollar bills [$f + 3 = o$, $5f + o = 75$]

30. Marilyn, 12 hours, Ruth, 6 hours [$\frac{1}{r} + \frac{1}{m} = \frac{1}{4}$, $\frac{3}{r} + \frac{6}{m} = 1$]

31. $\frac{3}{4}$[$\frac{n + 3}{d + 3} = \frac{6}{7}$, $\frac{n - 2}{d - 2} = \frac{1}{2}$]

32. '567' [$h + t + u = 3h + 3$, $u = h + 2$,
$100t + 10h + u = 7.3[(100t + 10h + u) - (100h + 10t + u)]$]

33. $23\frac{17}{21}$ feet [$\pi d(\frac{4}{5}d) = 500\pi = \pi(d - 4)h$]

Quiz items covering Unit 5. [These items differ somewhat from the exercises found in the textbook. You may want to use some of them along with others of your own design in a final examination.]

1. If 1 is a root of the quadratic equation in 'x':

$$5x^2 + tx + 1 = 0,$$

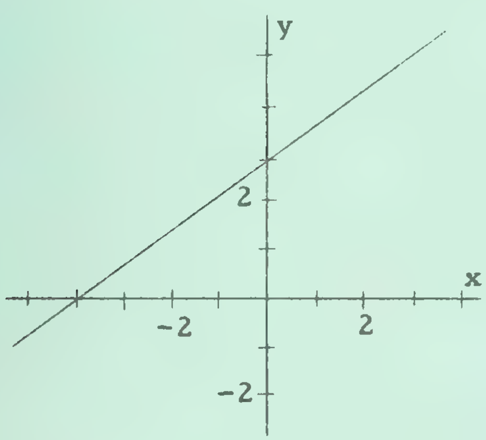
what is the other root?

2. Suppose f is the function such that

$$f(x) = \frac{(x - 3)(x + 5)}{(4x - 1)(x + 7)}.$$

For what real numbers is f not defined?

3.



$f = \{(x, y): y = \underline{\ ?\ }\ x + \underline{\ ?\ }\}$

4. The graph of the equation '$|x| + |y| = 10$' is __(?)__.

(A) a line (B) two lines (C) two rays

(D) four half-lines (E) a square

5. The dimensions of a 3-inch by 5-inch snapshot are doubled to make an enlargement. The area of the enlarged picture is __?__ times the area of the original snapshot.

6. Consider the set P of all linear functions which contain the ordered pair common to $\{(x, y): x - y - 2 = 0\}$ and $\{(x, y): 2x - y - 1 = 0\}$. Write the defining equation for the linear function in P which has slope 1.

7. Solve for 'M': $R = \frac{S(P - M)}{2s}$

8. If x and y are inversely proportional and if y is 30 when x is 5 then, when x is 25, y is _?_.

9. $\forall_x \forall_y$ if $x - y = 5$ and $x + y = 20$ then $x^2 - y^2 =$ _?_.

10. The number of ordered pairs which belong to both $\{(x, y): y = x^2\}$ and $\{(x, y): y = |x|\}$ is _?_.

11. A boy who has only k half-dollars and d dimes buys n notebooks at 15 cents each. How many cents does he have left?

12. Two variable quantities, P and Q, are so related that 5 times each value of P is twice the corresponding value of Q. Is P proportional to Q? If not, tell why. If so, give the factor of proportionality.

13. [Suppose that 'X′' denotes the complement of the set X with respect to some set S.] $(A \cup B)' =$ _(?)_.

(A) $A' \cup B'$ (B) $A' \cup B$ (C) $A \cup B'$

(D) $A \cap B$ (E) $A' \cap B'$

14. Suppose the sum of three consecutive integers is T. Then, the smallest of these integers is _?_.

15. Solve: $3x^2 - 2x = 2$

16. If the graphs of (2, 3), (4, 9), and (6, k) are all contained in a straight line then k = _?_.

17. Simplify: $\dfrac{1}{1 + \dfrac{1}{1 + a}}$

18. Consider the linear function f defined by the equation '$2x + 4y + 5 = 0$'.

(a) Find the slope of this function.

(b) Write an equation which defines the function whose graph is parallel to the graph of f and which contains (0, 0).

(c) Find the ordered pairs in the intersection of f and the x-axis.

19. Suppose that

$$f = \{(1, 2),\ (2, 2),\ (3, 0)\}$$

and $$g = \{(1, 3),\ (2, 3),\ (0, 1)\}.$$

Then, it follows that ___(?)___.

(A) $f \cup g$ is a function

(B) the domain of $f + g = \{1, 2, 3, 0\}$

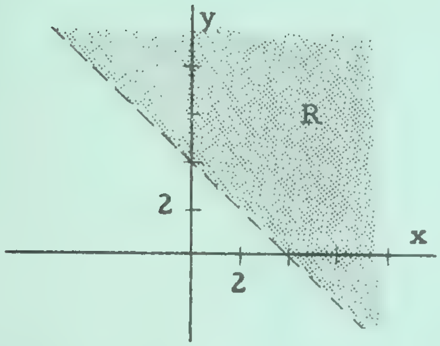
(C) the range of $f + g$ is $\{5\}$

(D) the range of $f + g$ is $\{2, 3\}$

(E) None of these follows.

20. If $(-1, 8)$ belongs to the quadratic function defined by '$y = 2x^2 + 4x + k$' then k = __?__.

21.



R = {(x, y): ______(?)______ }

(A) $x + y = 4$

(B) $x > 4$

(C) $x + y > 4$

(D) $x - y > 4$

(E) $x + y > 8$

22. If the slope of $\{(x, y):\ 2y - kx = 4\}$ is -3 then k = __?__.

23. $\{(x, y): x + y - 2 + |x + y - 2| = 0\} = \{(x, y): \underline{\quad (?) \quad}\}$

(A) $x = 1$ and $y = 1$ (B) $x + y - 2 > 0$ (C) $x + y - 2 = 0$

(D) $x + y - 2 < 0$ (E) $x + y - 2 \leq 0$

24. Two trains leave from the same city at the same time and travel in opposite directions. In 3 hours they are 300 miles apart. If one train's rate is 60 miles per hour then the other train's rate is __?__ miles per hour.

25. If the graphs of the equations '$x^2 + y^2 = 25$' and '$y = x^2$' are drawn on the same chart, what is the total number of points common to the graphs?

26. If the area-measure of a rectangle is $12x - x^2$ and x is the measure of a side then the area-measure is a maximum if $x =$ __?__.

27. Suppose the sets A and B are represented by circular regions. Then, the shaded region in Figure 1 represents $A \cap B$, and the

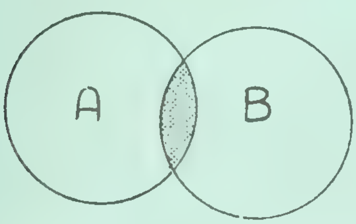


Figure 1.

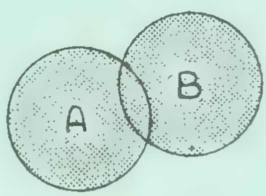


Figure 2.

shaded region in Figure 2 represents $A \cup B$. So, the shaded region in Figure 3 represents __(?)__.

(A) $A \cup (B \cap C)$

(B) $A \cap (B \cup C)$

(C) $(A \cap B) \cap C$

(D) $A \cup (B \cup C)$

(E) $(A \cup B) \cap C$

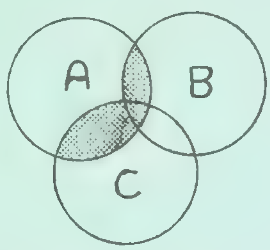


Figure 3.

28. Find the coordinates of the lowest point of the graph of the equation '$y = x^2 - 6x + 9$'.

29. True or false?

(a) $\forall_x \sqrt{x^2 + 9} = x + 3$ (b) $\exists_x \sqrt{x^2 + 9} = x + 3$

(c) $\forall_x x^2 + 1 = 0$ (d) $\exists_x x^2 + 1 = 0$

(e) $\forall_x x^2 - 8x + 16 > 0$ (f) $\exists_x x^2 - 8x + 16 > 0$

30. Suppose that $f = \{(1, 2), (3, 4), (5, 6), (7, 8)\}$. Then, the inverse of f is ___(?)___.

(A) $\{(7, 8), (5, 6), (3, 4), (1, 2)\}$

(B) $\{(1, -2), (3, -4), (5, -6), (7, -8)\}$

(C) $\{(2, 1), (4, 3), (6, 5), (8, 7)\}$

(D) $\{(1, \frac{1}{2}), (3, \frac{1}{4}), (5, \frac{1}{2}), (7, \frac{1}{8})\}$

(E) nonexistent

31. If $(2, -1) \in \{(x, y): x = ky + t\} \cap \{(x, y): 3x + ky = t\}$ then $k =$ ___?___ and $t =$ ___?___.

32. The roots of '$(m - 4)(m^2 + 3m - 10) = 0$' are ___?___, ___?___, and ___?___.

33. A quadratic function has 2 as maximum value. If it contains the ordered pairs $(-\frac{1}{2}, 0)$ and $(\frac{3}{2}, 0)$ then its defining equation is: $y =$ ___?___ $x^2 +$ ___?___ $x +$ ___?___.

34. The graph of '$x^2 + 2xy + y^2 = 1$' is ___(?)___.

(A) a straight line (B) two straight lines (C) a circle

(D) a parabola (E) an hyperbola

35. The sum of the area-measures of a rectangle and a square is 68. The rectangle is twice as long as it is wide, and a side of the square is 2 units longer than the width of the rectangle. Find the width-measure and the length-measure of the rectangle.

36. What is the solution set in x of '$4 - 2x < 10$'? [Use brace-notation and the simplest set selector to describe the solution set.]

37. Ten quarts of a solution containing m% alcohol are mixed with twenty quarts of another solution containing n% alcohol. The part of the resulting mixture which is alcohol is ___?___ .

38. Bill travels from A to B, a trip of 30 miles, at the rate of 6 miles per hour, and returns immediately from B to A. If his average rate for the entire trip was 5 miles per hour then his return rate was ___?___ miles per hour.

39. Consider the equation in 'x':

$$x^3 + px + q = 0$$

If two of its roots are -1 and 3 then $p =$ ___?___ and $q =$ ___?___ .

40. If the quadratic equation '$x^2 + 6x - 2 + k = 0$' has just one root then $k =$ ___?___ .

41. Write a quadratic equation whose solution set is $\{\frac{1}{3}, -3\}$.

42. The graphs of the equations '$3x + 5y = 9$', '$6x - 10y = 10$', and '$6x + 10y = 11$' are ___(?)___ .

(A) three parallel lines [///]

(B) two parallel lines crossed by a third line [ ]

(C) three lines no two of which are parallel but with no point common to all three [ ]

(D) three lines intersecting in a single point [ ]

(E) two lines only

43. If y varies inversely as x then the graph obtained by plotting y against x is __(?)__.

(A) a straight line (B) a circle (C) an hyperbola

(D) a parabola (E) an ellipse

44. Suppose that $f = \{(1, 2), (2, 3), (3, 4), (5, 6)\}$. Which of the following statements is true?

(A) the domain of $f = \{1, 2, 3, 4, 5\}$ (B) the range of $f = \{2, 3, 4, 6\}$

(C) $f(2) = 1$ (D) $f(f(2)) = 2$ (E) None of these is true.

45. The area-measure of a circle varies directly as the square of its radius. The factor of variation is __?__.

46. Suppose f is the function such that $f(x) = \sqrt{2x + 3}$. The domain of $f = \{x: \underline{\quad ? \quad}\}$

47. The expression '$x^2 + \frac{k}{n}x + \underline{\ ?\ }$' is a perfect square.

48. Suppose that $t > 0$ and $2t^2 + 5t - 33 = 0$. Then, $t = \underline{\ ?\ }$.

49. If x varies directly as y and if $x = 12$ when $y = 8$ then $x = \underline{\ ?\ }$ when $y = 10$.

50. Consider a quadratic equation in 'x' of the form:

$$qx^2 + sx + t = 0, \qquad [q > 0]$$

Suppose the roots are r_1 and r_2.

(a) If $r_1 < 0$ and $r_2 > 0$ then __(?)__.

(A) $s > 0$ (B) $s < 0$ (C) $t > 0$ (D) $t < 0$

(b) If $r_1 = r_2$ then __(?)__.

(E) $s^2 = 4qt$ (F) $s^2 = -4qt$ (G) $s^2 = qt$ (H) $s^2 = -qt$

51. Write an equation whose straight line graph contains the graph of (0, −3) and is parallel to the graph of 'y = 2x + 6'.

52. Suppose that D is the doubling function and S is the squaring function.

(a) $S(-5) =$ ____ (b) $D(-5) =$ ____ (c) $S(0) =$ ____

(d) $D(0) =$ ____ (e) $D^2(3) =$ ____ (f) $S^2(3) =$ ____

(g) $[D \circ S](5) =$ ____ (h) $[S \circ D](5) =$ ____ (i) $D^{-1}(9) =$ ____

*

Answers for quiz items.

1. $\frac{1}{5}$ 2. $\frac{1}{4}$ and -7 3. $\frac{3}{4}$, 3 4. (E) 5. 4

6. $y = x - 2$ 7. $M = \frac{PS - 2sR}{S}$ 8. 6 9. 100

10. 3 11. $50k + 10d - 15n$ 12. Yes; 2/5

13. (E) 14. $\frac{T - 3}{3}$ 15. $\frac{1 + \sqrt{7}}{3}, \frac{1 - \sqrt{7}}{3}$ 16. 15

17. $\frac{1 + a}{2 + a}$ 18. (a) $-\frac{1}{2}$ (b) $y = -\frac{1}{2}x$ (c) $(-\frac{5}{2}, 0)$ 19. (C)

20. 10 21. (C) 22. −6 23. (E) 24. 40

25. 2 26 6 27. (B) 28. (3, 0)

29. (a) F (b) T (c) F (d) F (e) F (f) T

30. (C) 31. 2, 4 32. 4, −5, 2 33. −2, 2, 3/2

34. (B) 35. 4, 8 36. $\{x: x > -3\}$

37. $\frac{m + 2n}{300}$ 38. 30/7 39. −7, −6 40. 11

41. $3x^2 + 8x - 3 = 0$ 42. (B) 43. (C) 44. (B)

45. π 46. $x \geq -\frac{3}{2}$ 47. $\left(\frac{k}{2n}\right)^2$ 48. 3

49. 15 50. (a) (D) (b) (E) 51. $y = 2x - 3$

52. (a) 25 (b) -10 (c) 0 (d) 0 (e) 36

(f) 81 (g) 50 (h) 100 (i) 4.5

Answers for MISCELLANEOUS EXERCISES.

A.	1.	$4x^2 - 9$	2.	$8x$	3.	$3a + 7b$
	4.	$3x^3$	5.	$7x/6$	6.	$7/6$
	7.	$-7\sqrt{2}$	8.	$2k$	9.	$2y^2$
	10.	$3x^2 - 7x - 20$	11.	$m - 5n$	12.	$7a + 6b + 9$
	13.	$23x/12$	14.	$x - 3$	15.	$4x^4y$
	16.	$8b + 1$	17.	$4a^2 + 17a + 4$	18.	$16\sqrt{2}$
	19.	$6k^2 - 19k + 10$	20.	$10y$	21.	$2x^4$
	22.	$49s^2 - t^2$	23.	3	24.	$x - 2y$
	25.	$4\sqrt{3}$	26.	$-x^2 + 7x - 3$	27.	$(x + 5)/5$
	28.	$(x + 5y)/6$	29.	$x^2 + 2x + 8$	30.	$m^2 - 7m + 10$
	31.	$16 - x^2$	32.	13		

*

B.	1.	$3m(k + 5)$	2.	$5k^4m(1 + 14k^2)$	3.	$4r(x + 3)$
	4.	$7x^3y(x^2 + 3)$	5.	$(c - 5)(c + 5)$	6.	$(a - 6)(a + 6)$
	7.	$5a^2b^2(a + 3)$	8.	$4a^3b^3(a^2 + 2)$	9.	$(3x - 1)(3x + 1)$
	10.	$(x - 7)(x + 7)$	11.	$3rs^2(rs + 2)$	12.	$4ab(b + 5)$

*

C.	1.	5	2.	4	3.	3, -3	4.	-6	5.	1
	6.	12	7.	12	8.	5	9.	4	10.	10
	11.	6	12.	6	13.	7	14.	3	15.	5
	16.	no roots	17.	14	18.	1	19.	2.5	20.	5
	21.	-1	22.	1	23.	4	24.	4.5	25.	6

	26.	9	27.	3	28.	$\{x:\ x < 3\}$
	29.	$\{x:\ x < 6\}$	30.	6	31.	7
	32.	$\{x:\ x > -3\}$	33.	$\{x:\ x > 12\}$	34.	$\{y:\ y > 7\}$

*

D. 1. $\frac{m}{k}$ dollars 2. $x + 3$ 3. 80 4. (c) [about 499]

5. (0, 2) 6. -11 7. $1/(x - 3)$

*

E. 1. $x^2 - 5x - 36$ 2. $x^2 - 10x + 16$ 3. $x^2 + x - 56$
4. $x^2 + 8x + 16$ 5. $x^2 - 10x + 25$ 6. $9x^2 - 6x + 1$
7. $25 - 20y + 4y^2$ 8. $15 - 2x - x^2$ 9. $6x^2 + 13x - 28$
10. $a^4 - 25$ 11. $m^4 - n^4$ 12. $3x^2 + 39x - 204$
13. $4x^2 - 4x - 15$ 14. $4Q^2 - 4QA + A^2$ 15. $A^2 - 4AQ + 4Q^2$

*

F. 1. $(x + 7)(x + 9)$ 2. $(m - 11)(m + 2)$ 3. $(x - 19)(x + 19)$
4. $(18 - p)(1 - p)$ 5. $(k + 72)(k - 2)$ 6. $(3 - y)(3 + y)(9 + y^2)$
7. $(D + E)^2$ 8. $2(x + 3)(x + 5)$ 9. $7(y - 3)(y + 2)$
10. $5(A - 3)(A - 8)$ 11. $(3F - 2N)(2F - 3N)$ 12. $(d + 13)(d - 7)$

*

G. 1. 0, 7 2. 10, 20 3. 3, -17 4. $\{x\colon x < 2 \text{ or } x > 3\}$
5. all real numbers 6. no roots

*

H. 1. F [-1] 2. F [-1] 3. F [$1/3$] 4. F [-1] 5. F ['and']
6. F [$1.414^2 < 2$] 7. F [$1/2$] 8. F 9. T

*

I. 1. $1\frac{3}{4}$ 2. $n + 2$ 3. 5 4. $x/15$ 5. $x = (y-2)/$
6. 75 7. $2x - 3$ 8. $(x - 2y)/2$ 9. $5\sqrt{3}$ [$\sqrt{75} > \sqrt{45}$]

*

J. 1. T 2. T 3. T 4. T 5. T
6. F[$n(A) = 27$, $n(B) = 3$] 7. F[$n(A) = 1$, $n(B) = 49$]

8. T   9. F [$A = \emptyset \neq B$]

10. T [$A \cup (B \cup C) = (A \cup A) \cup (B \cup C)$. Then, by commutativity and associativity, it follows that $(A \cup A) \cup (B \cup C) = (A \cup B) \cup (A \cup C)$.]

*

K. 1. $7x$ 2. T 3. 7[neglecting sawcuts] 4. $2x/3$ dollars 5. $3t + 2$
6. 1 7. $4.20 8. -2 9. $3c$ 10. $2x - 2y$

*

L. 1. two 2. one 3. one 4. 0 5. 25
6. 9 7. 144 8. 81 9. 36 10. 30
11. 2 12. 783 13. 5163 14. 835 15. 589
16. 347, 698 17. 1873, 6952 18. 1873, 6952
19. 1873, 6952 20. 189, 189 21. 64, 36
22. 64, 36 23. 5, 7 24. 5, 7 25. one 26. 5, 7
27. 18, 2 28. p 29. ab 30. ab 31. ab
32. ab 33. ab 34. one, $\sqrt{ab}$ 35. 25, 3, 5, 3
36. 500, 100, 10 37. 49, 2, 49, 2, 7, 2
38. 28, 4, 7, 4, 7 39. 99, 9, 11, 9, 11
40. 16, 10, 16, 10, 4 41. 36, 3, 36, 3, 3
42. 9, 7, 3, 7 43. 363 44. 32

*

M. 1. $\frac{\sqrt{2}}{4}$ 2. $\frac{3 + \sqrt{3}}{2}$ 3. $\frac{\sqrt{3}}{2}$ 4. $\frac{\sqrt{6}}{4}$

5. $\frac{\sqrt{3}}{4}$ 6. $\frac{2\sqrt{2}}{11}$ 7. $8 + \sqrt{3}$ 8. $7 - \sqrt{7}$

9. $\frac{12 + \sqrt{7}}{5}$ 10. $\frac{-1 - \sqrt{22}}{3}$ 11. 1 12. $\frac{6 - \sqrt{2}}{2}$

13. $5\sqrt{2}$ 14. $5\sqrt{2}$ 15. $8\sqrt{3}$ 16. $2\sqrt{5}$

17. 2 18. 2 19. 1 20. 5

21. 7 22. 6 23. $\sqrt{6}$ 24. 5

25. $\sqrt{5}$ 26. 11 27. $\sqrt{11}$ 28. $\sqrt{3}$

29. $\sqrt{3}$ 30. a, a, 1, $\sqrt{b}$, $\sqrt{a}$, $\sqrt{b}$ 31. $2\sqrt{5}$

32. $\frac{\sqrt{3}}{2}$ 33. $\frac{12}{13}$ 34. $\frac{\sqrt{7}}{3}$ 35. 3

36. $\frac{\sqrt{3}}{3}$ 37. $4\sqrt{5}$ 38. $\sqrt{3}$ 39. $\frac{3\sqrt{7}}{2}$

40. $\frac{3\sqrt{3}}{5}$ 41. $\frac{\sqrt{3}}{6}$ 42. $\sqrt{2} + \sqrt{5}$ 43. $16\sqrt{2} + 8\sqrt{5}$

44. 60 45. $\sqrt{1817}$ 46. $7\sqrt{2}$ 47. $7\sqrt{3}$

*

N. 1. $\frac{7 + 3\sqrt{5}}{2}$ 2. $\frac{14 + 5\sqrt{3}}{2}$ 3. $\frac{5 - 2\sqrt{6}}{2}$ 4. $\frac{8 + 3\sqrt{7}}{8}$

5. $\frac{33 + 12\sqrt{6}}{4}$ 6. $\frac{33 - 12\sqrt{6}}{4}$ 7. $\frac{13 + 4\sqrt{3}}{4}$

8. Yes. $[(1 + \sqrt{6})^2 - 2(1 + \sqrt{6}) - 5 = 7 + 2\sqrt{6} - 2 - 2\sqrt{6} - 5 = 0]$

9. $2\left(\frac{5 + \sqrt{89}}{4}\right)^2 - 5\left(\frac{5 + \sqrt{89}}{4}\right) - 8$

$= \frac{114 + 10\sqrt{89}}{8} - \frac{50 + 10\sqrt{89}}{8} - \frac{64}{8} = \frac{0}{8} = 0$

$2\left(\frac{5 - \sqrt{89}}{2}\right)^2 - 5\left(\frac{5 - \sqrt{89}}{4}\right) - 8$

$= \frac{114 - 10\sqrt{89}}{8} - \frac{50 - 10\sqrt{89}}{8} - \frac{64}{8} = \frac{0}{8} = 0$

O. 1. 22 2. 50 3. 150 4. 150

5. 80 6. 150 7. 1.5 H 8. 100 + x

9. m + 0.01 mn [or: m(1 + .01n)] 10. 12.5a

11. 3600/d 12. 4 13. 30 14. 100f/g

15. 0.01jk 16. 100p/r 17. 120 18. 25

19.

List price	50	72	100	170	40	78	60	320
Discount rate	40%	$12\frac{1}{2}$%	25%	30%	$62\frac{1}{2}$%	$37\frac{1}{2}$%	$3\frac{1}{3}$%	$6\frac{1}{4}$%
Purchase price	30	63	75	119	15	48.75	58	300

20. [If the discount rate were 22%, he would pay $5460.]

(a) 63 (b) 380 (c) 380 (d) 802.20

(e) 2730 (f) 600 (g) 680 (h) 30

21. $2000 22. $24 23. He lost $5.

*

P. [Students should be encouraged to consult a dictionary for the meanings of terms which may be new to them.]

1. 200 2. 12 3. 6 4. 160

5. 3 6. 198 7. 7.62 8. 100

9. 30.48 10. 39.37* 11. 0.9144 12. 100000

13. 1.609344 14. 0.3281* 15. 3600 16. 1500

17. 0.2 18. 0.35 19. 91.44 20. 5

21. 0.5 22. 0.3048 23. 0.01 24. 144

25. 0.5 26. 6.4516 27. 495 28. 6.665*

29. 0.0278 30. 200 31. 2700 32. 2

33. 16387.064 34. 5451776000 35. 0.000062 36. d/7

37. 16p 38. p/2000 39. s/3600 40. 63360m

*This is an approximation based on the fact that 1 centimeter is approximately 0.3937 inches. The United States standard inch is defined to be 2.54 centimeters.

Q. 1. 34 2. 76597 3. 68086; 99 4. 9 5. 12
6. 54 7. 77 8. 99 9. same 10. Yes

*

R. 1. (a) 3 (b) 0 (c) 31 (d) 23 (e) 11
(f) 11 (g) 20 (h) 20 (i) 20 (j) 12
(k) 19 (l) 19 (m) 17 (n) 17 (o) $16\frac{23}{31}$

2. (a) 7; no (b) 6 (c) $6\frac{7}{8}$

3. (a) 128 (b) $127\frac{9}{11}$

*

S. 1. 2 and 5 2. 35.5 and 40.5 3. 4 and 15
4. 2, 8; −8, −2 ☆5. 9 and 6 6. Al is 24 and Bill is 8.
7. Kathy is 16 years 4 months and Cheryl is 14 years 8 months.
8. A half-dollar and a dime [Although one of the coins is not a dime, the other is.]
9. $68.25 10. 24 dimes, 96 nickels, 72 pennies
11. 2 12. 2 13. 33; 60; 55; 45
14. 2/3 gallon 15. 0.9 gallon 16. 1/16 quart
17. 12.5 gallons 18. 0.2 gallon 19. [can't be done]
20. $71.24 21. y/x cents; wy/x cents 22. $8\frac{1}{3}$
23. $53\frac{11}{13}$ pounds of creams

24. (a) 2 mph | (b) $1\frac{1}{3}$ mph | 25. 3 hours

T.

1. $y = -3x + 7$
2. $y = 5x + 9$
3. $y = 4x + 6$
4. $y = -2x + \frac{10}{3}$
5. $b = -\frac{2}{5}a + \frac{7}{5}$
6. $x = \frac{s}{t - n}$
7. $n = 3m - 8$
8. $m = \frac{1}{3}n + \frac{8}{3}$
9. $y = -ax - c$
10. $y = -\frac{a}{b}x - \frac{c}{b}$
11. $y = \frac{k}{m + n}$
12. $s = -\frac{10}{7}r + \frac{5}{7}$
13. $y = 2z + 1$
14. $y = 4x - \frac{5}{3}$
15. $y = -\frac{5}{7}x + \frac{43}{7}$
16. $b = -9a + 52$
17. $p = k + m$
18. $x = t - s$
19. $x = \frac{3}{5}y$
20. $y = \frac{13}{14}$

U.

1. $2a + 7$
2. $2x + 7$
3. $2k - m$
4. $x + 5y$
5. $8x^2 - 6x$
6. $10a^2 - 2b^2$
7. $7ab - 8bc$
8. $16xy - 3x - 7y$
9. $6m^2 + mn - 3n$
10. $4p^2 - 3pq + 3q^2$
11. $-3a - 24b - 30$
12. $19x - 8y - 26$
13. $2m + 15n$
14. $3x - 2y + 13z$
15. $8x^2 - 19x$
16. $-14y^2 - 3$

17. $2x^3 - x^2 + 3x$

18. $-3x^2 - 34xy + 15y^2 - 5y$

19. $6a^2 - 13ab + 14b^2$

20. $7x^4 - 16x^3 + 17x^2$

21. $-30abc$

22. $6xyz$

23. $-70abm$

24. $288x^3y^2$

25. $0.88a^3$

26. $6k^4$

27. $-24a^9$

28. $18b^6$

29. $6x^4y^4$

30. $-6a^4b^7$

31. $360m^8n^8$

32. $24a^{13}b^9$

33. $3x^3y^3$

34. $2a^3b^4$

35. $30a^2b^2$

36. $13x^2y^4 - 17x^3y^7$

37. $\frac{1}{2}$

38. $\frac{x}{y}$

39. $\frac{x}{y}$

40. c

41. $\frac{5}{x^2}$

42. $\frac{1}{3xy}$

43. $\frac{x(a+b)}{3}$

44. $c - d$

45. $\frac{11x + 30}{15x}$

46. $\frac{x^2 + 2x - 12}{4x}$

47. $\frac{3xy - 28}{6x}$

48. $\frac{2 - 3x + 4y}{xy}$

49. $\frac{6 - 9x}{2x^2}$

50. $\frac{2xy - 3x + y}{x^2y^2}$

51. $\frac{7x + 19}{(x+3)(x+2)}$

52. $\frac{7(y+2)}{(y-2)(y+5)}$

53. $\frac{3z - 11}{(z-1)(z-2)}$

54. $\frac{5x + 16}{(x+3)^2}$

55. $\frac{3x - 2}{(x+4)(x-1)}$

56. $\frac{3x - 8}{(x-5)(x+1)}$

57. $\frac{8x + 23}{(x+2)(x+3)}$

58. $\frac{2(2y + 21)}{y^2 - 16}$

59. $\frac{3a^2 + 9ab + 8b^2 - 6a + 5b}{(a-b)(a+3b)}$

60. $\frac{z + 18}{(z-3)(z+5)}$

61. $\frac{15ab^2y}{14x}$

62. $\frac{b^2x}{6a^4y}$

63. $\frac{7 - y}{9(x-5)^2}$

64. $\frac{(x-2)(y+5)}{(x-4)(y+7)}$

65. $\frac{(x-2)(y+5)}{(x-4)(y+7)}$

66. $\frac{(a+2)(b+6)}{(a+4)(b-3)}$

67. $\frac{a^2(a-7)(a+5)}{a^3 - 2a^2 - 35}$

Answers for SUPPLEMENTARY EXERCISES.

A.

Exercises 7 and 8 should be assigned together. Each of these relations is the complement of the other with respect to the number plane [that is, with respect to the cartesian square of the set of real numbers]. Note that each of the set selectors is equivalent to the denial of the other. The denial of '$a \geq 2$ or $b \leq 3$' [Exercise 7] is 'not ($a \geq 2$ or $b \leq 3$)' [or, in better English, 'neither $a \geq 2$ nor $b \leq 3$'], and this is equivalent to 'not $a \geq 2$ and not $b \leq 3$' [Exercise 8]. In general, the denial of each alternation sentence is equivalent to a conjunction sentence--that is, each sentence of the form 'not (p or q)' is equivalent to the corresponding sentence of the form 'not p and not q'. Similarly, each sentence of the form 'not (p and q)' is equivalent to the corresponding sentence of the form 'not p or not q'--the denial of each conjunction sentence is equivalent to an alternation sentence. So, the denial, 'not ($a \not\geq 2$ and $b \not\leq 3$)', of the set selector of Exercise 8 is equivalent to 'not $a \not\geq 2$ or not $b \not\leq 3$' and, so, to '$a \geq 2$ or $b \leq 3$'. [This last transformation depends on the fact that replacing 'p' in 'not not p' by a sentence yields a sentence which is equivalent to the replacement.]

These three logical principles concerning denials of alternations, denials of conjunctions, and denials of denials are analogous to DeMorgan's laws and the principle that $\forall_x \,\tilde{\tilde{x}} = x$, all mentioned on pages 5-22 and 5-23. The three theorems for sets become obviously applicable if we note that

[Ex. 7] $\{(a, b)\colon\ a \geq 2 \text{ or } b \leq 3\} = \{(a, b)\colon\ a \geq 2\} \cup \{(a, b)\colon\ b \leq 3\}$,

and that

[Ex. 8] $\{(a, b)\colon\ a \not\geq 2 \text{ and } b \not\leq 3\} = \{(a, b)\colon\ a \not\geq 2\} \cap \{(a, b)\colon\ b \not\leq 3\}$,

and that

$$\{(a, b)\colon\ a \not\geq 2\} = \widetilde{\{(a, b)\colon\ a \geq 2\}}$$

and

$$\{(a, b)\colon\ b \not\leq 3\} = \widetilde{\{(a, b)\colon\ b \leq 3\}}.$$

In Exercise 9, the index '(x, y), $x \neq 0$' in the brace-notation name tells you that the relation in question is a subset of the set of all ordered pairs of real numbers whose first component is not 0. So, strictly speaking, the graph of the y-axis ought to be eliminated from the picture, perhaps by using a dashed line as we do. The important fact to be noted by the students is that the relation is not a straight line but rather the union of two disjoint and collinear half-lines with a common end point which belongs to neither of them. They should show this on the picture by placing a hollow dot at the graph of the common end point.

Exercise 13 provides an interesting graph. One part of it is a series of discrete dots, and the other is a series of discrete intervals.

The relation in Exercise 11 is the complement of a pair of parallel lines.

Students may like to know that the relation in Exercise 16 is an hyperbola [specifically, an equilateral hyperbola, one whose asymptotes are perpendicular]. [See page 5-155.]

The relation in Exercise 17 is the empty set. There is no ordered pair whose second component is such that the sum of its absolute value and a number greater than 5 is 3.

Students may get a clue for Exercise 19 from Exercise 18. The trick in Exercise 19 is to factor the left member of the set selector.

$$xy - x^2 = 0$$

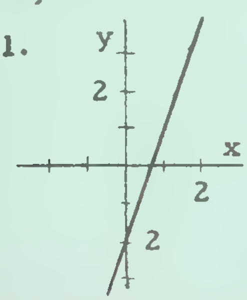
$$x(y - x) = 0$$

$$x = 0 \text{ or } y - x = 0$$

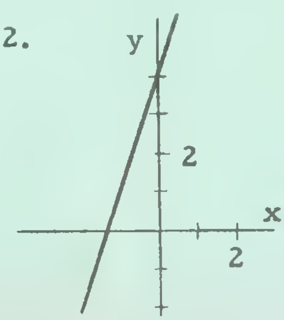
The last sentence is equivalent to the first. The locus is the union of the y-axis and the locus of '$y = x$'.

Students will probably first find the boundary of the graph for Exercise 20 by graphing the equation: $x^2 - 7x + 10 = y$. Then, for each point on the boundary, this point and all points with the same x-coordinate and a larger y-coordinate are in the graph.

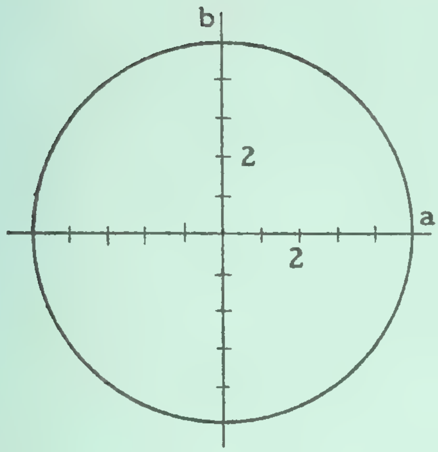
1.



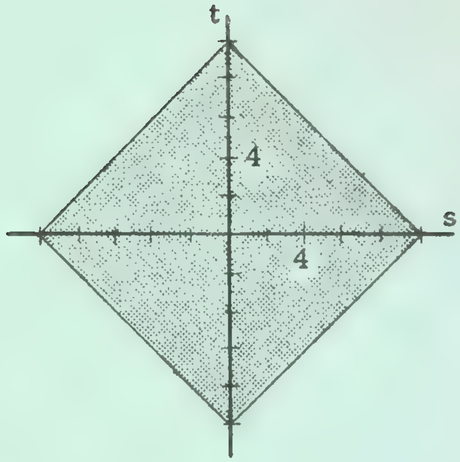
2.



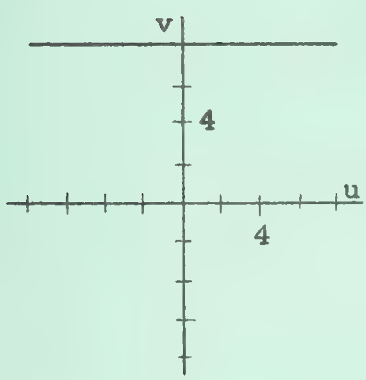
3.



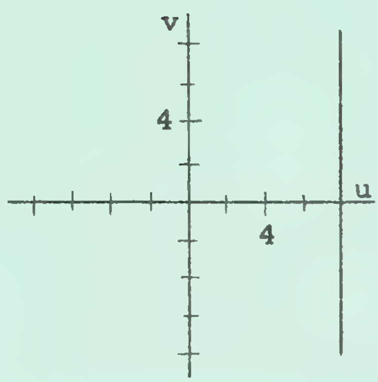
4.



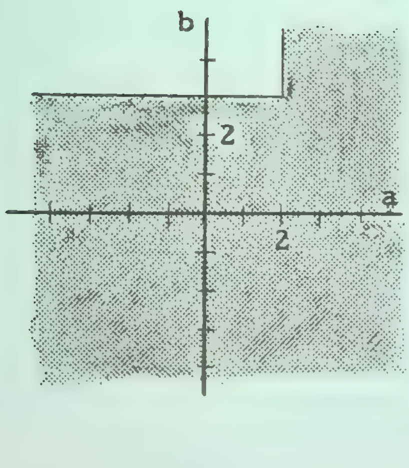
5.



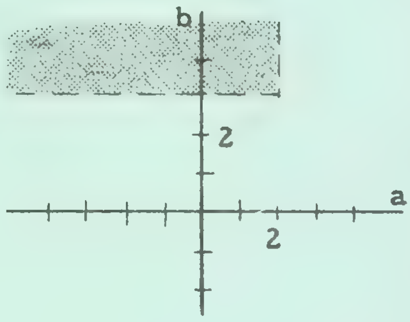
6.



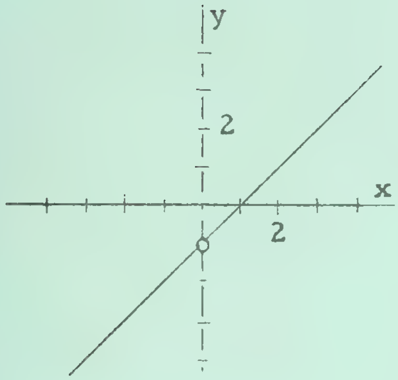
7.



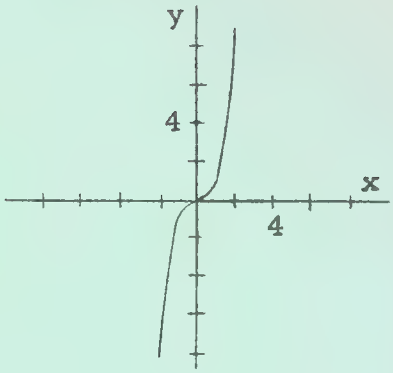
8.



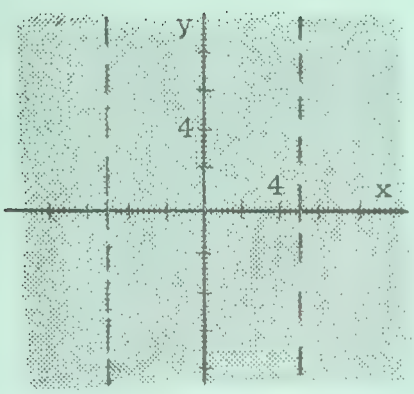
9.



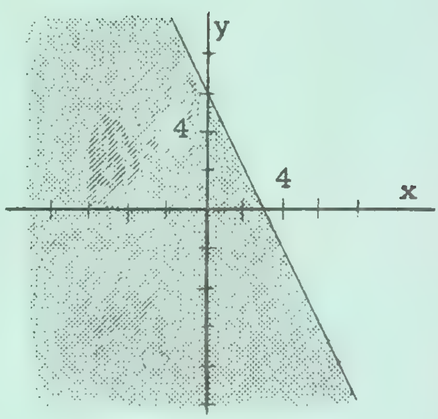
10.



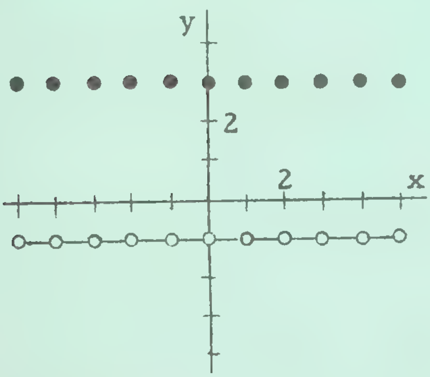
11.



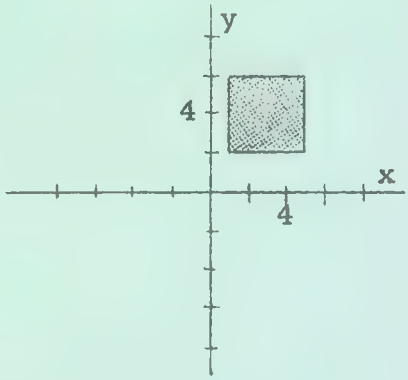
12.



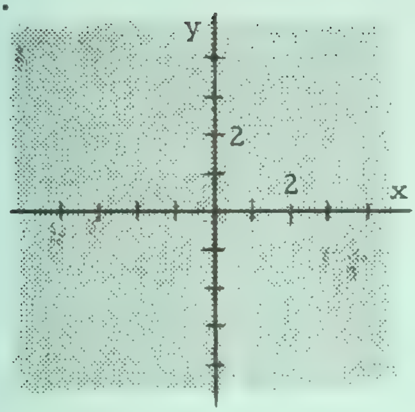
13.



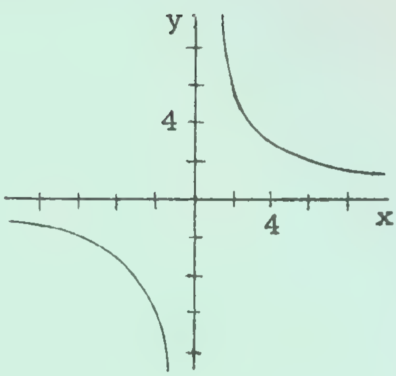
14.



15.

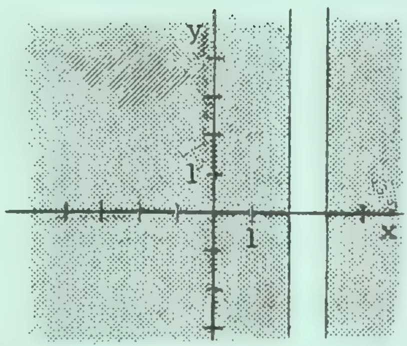


16.

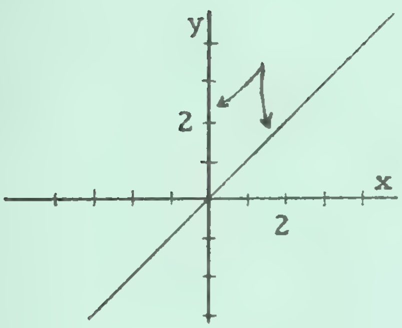


17. The relation is empty. So, there are no points to graph.

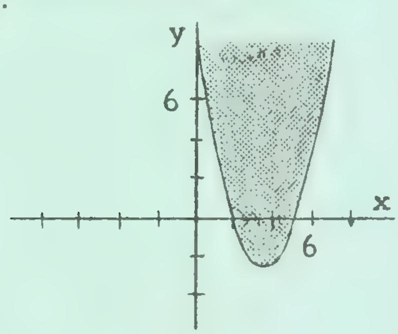
☆18.



☆19.



☆20.



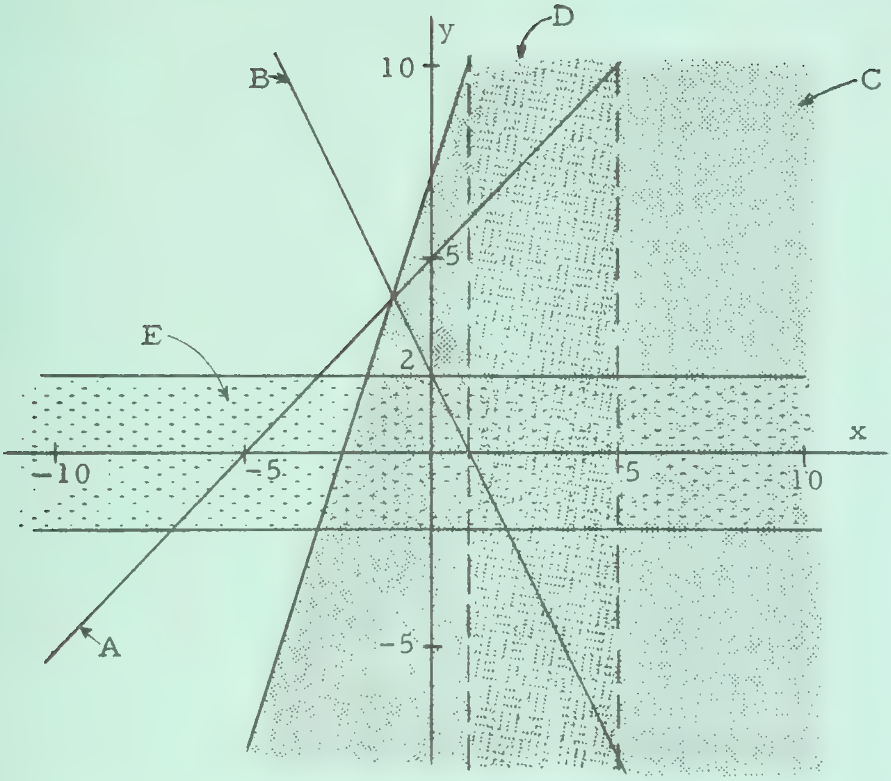
*

B. [on pages 5-238 and 5-239]

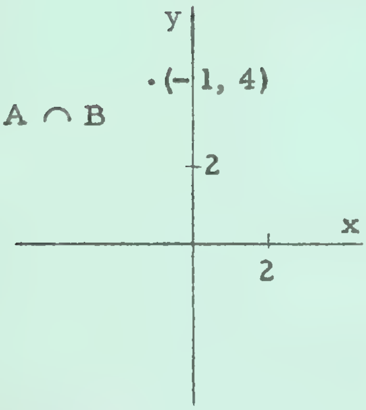
1. (c), (e) 2. (b), (d) 3. (a) 4. (b), (d)

5. (b), (c), (e) 6. (b), (c), (e) 7. (c)

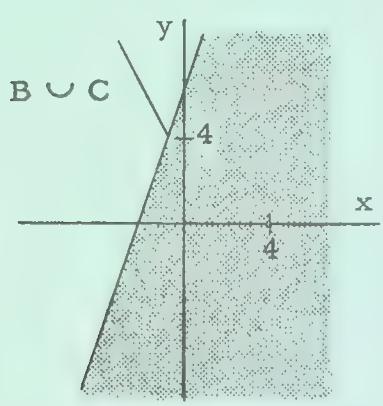
C. 1.



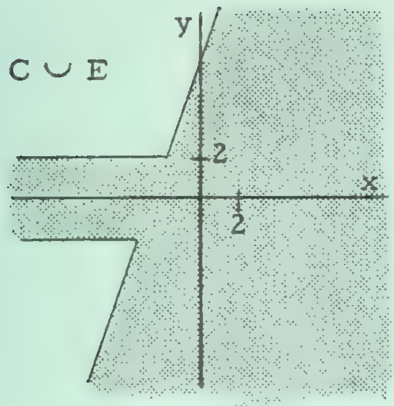
2.



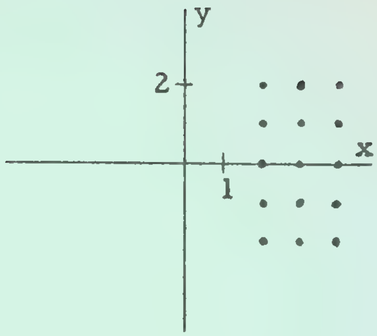
3.



4.

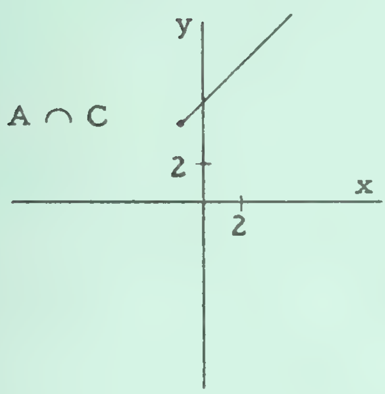


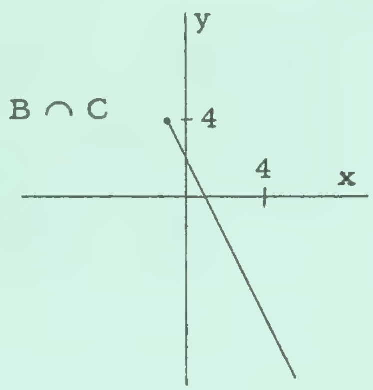
5.

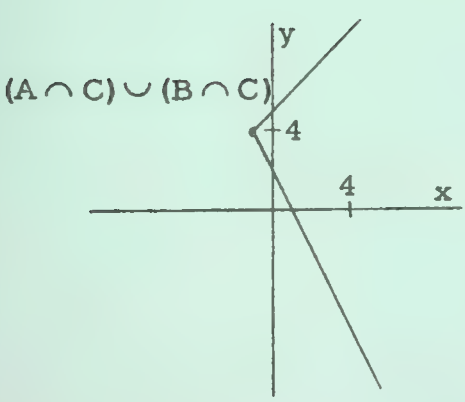


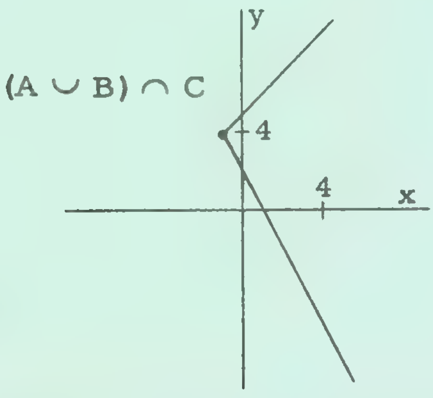
(2, 2), (2, 1), (2, 0), (2, −1), (2, −2)
(3, 2), (3, 1), (3, 0), (3, −1), (3, −2)
(4, 2), (4, 1), (4, 0), (4, −1), (4, −2)

6.



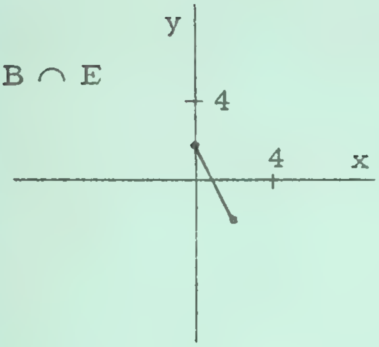




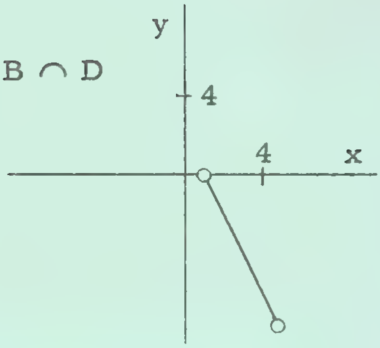


7.

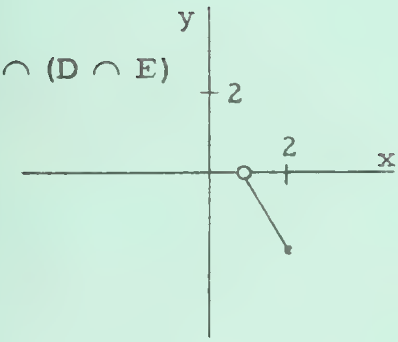
$B \cap E$



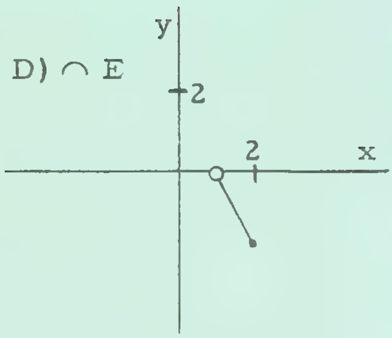
$B \cap D$



$B \cap (D \cap E)$

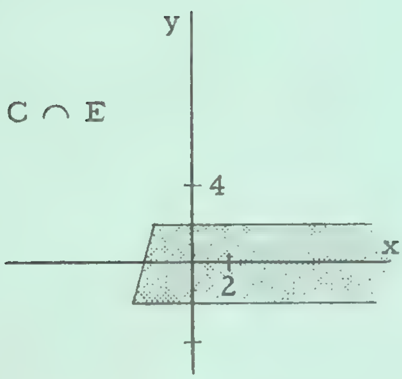


$(B \cap D) \cap E$

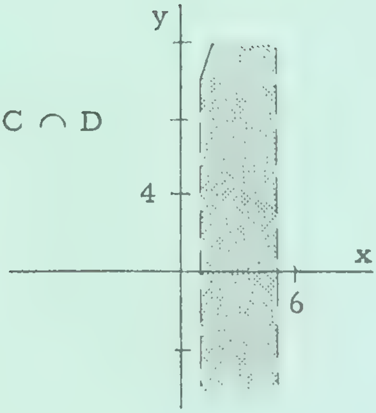


8.

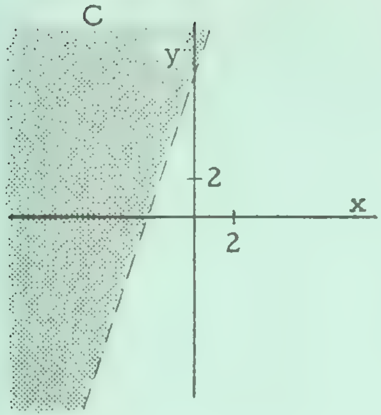
$C \cap E$



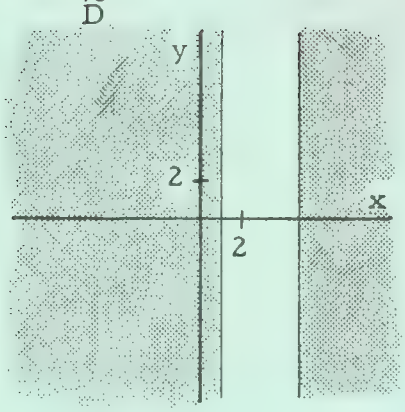
$C \cap D$



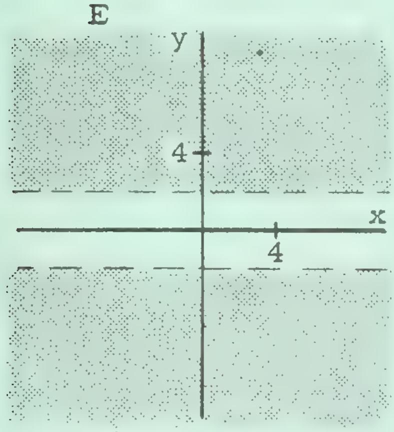
9. $\tilde{C}$



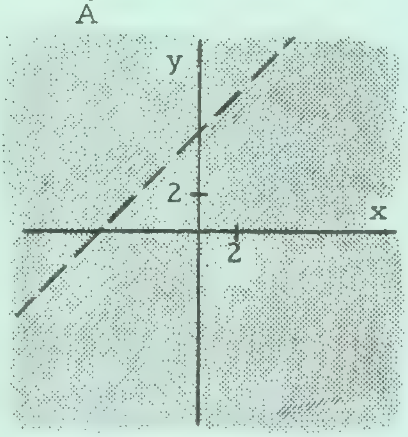
$\tilde{D}$



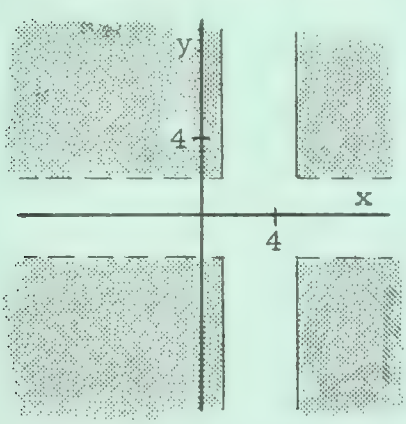
$\tilde{E}$



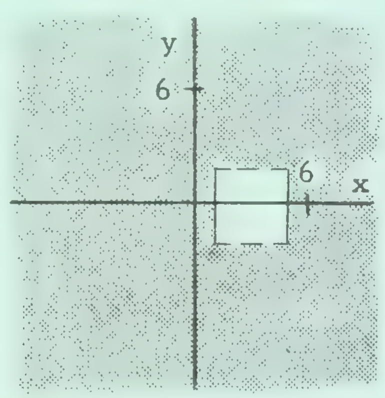
$\tilde{A}$



10.



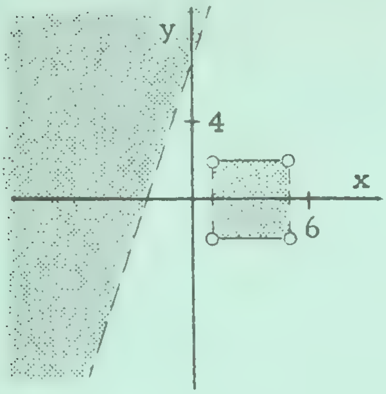
[$\tilde{D} \cap \tilde{E} = \widetilde{D \cup E}$]



[$\tilde{D} \cup \tilde{E} = \widetilde{D \cap E}$]

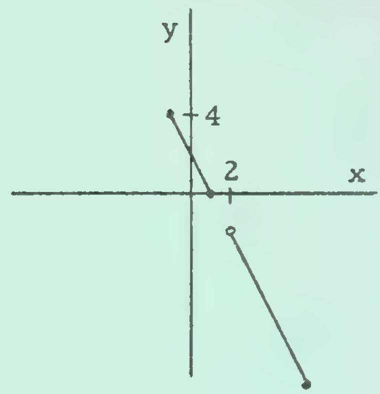
11.

$(D \cap E) \cup \tilde{C}$

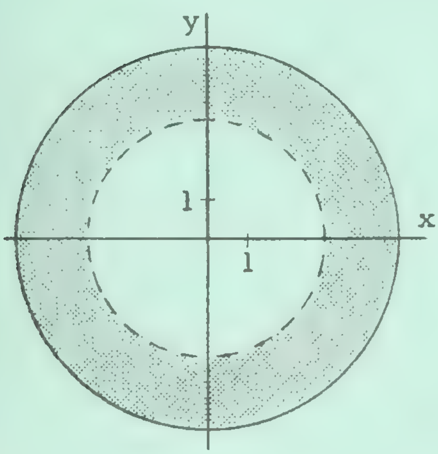


12.

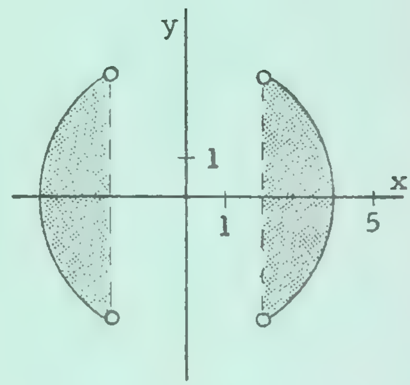
$[C \cap \widetilde{(D \cap E)}] \cap B$



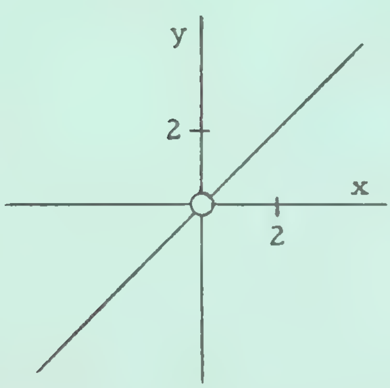
13.



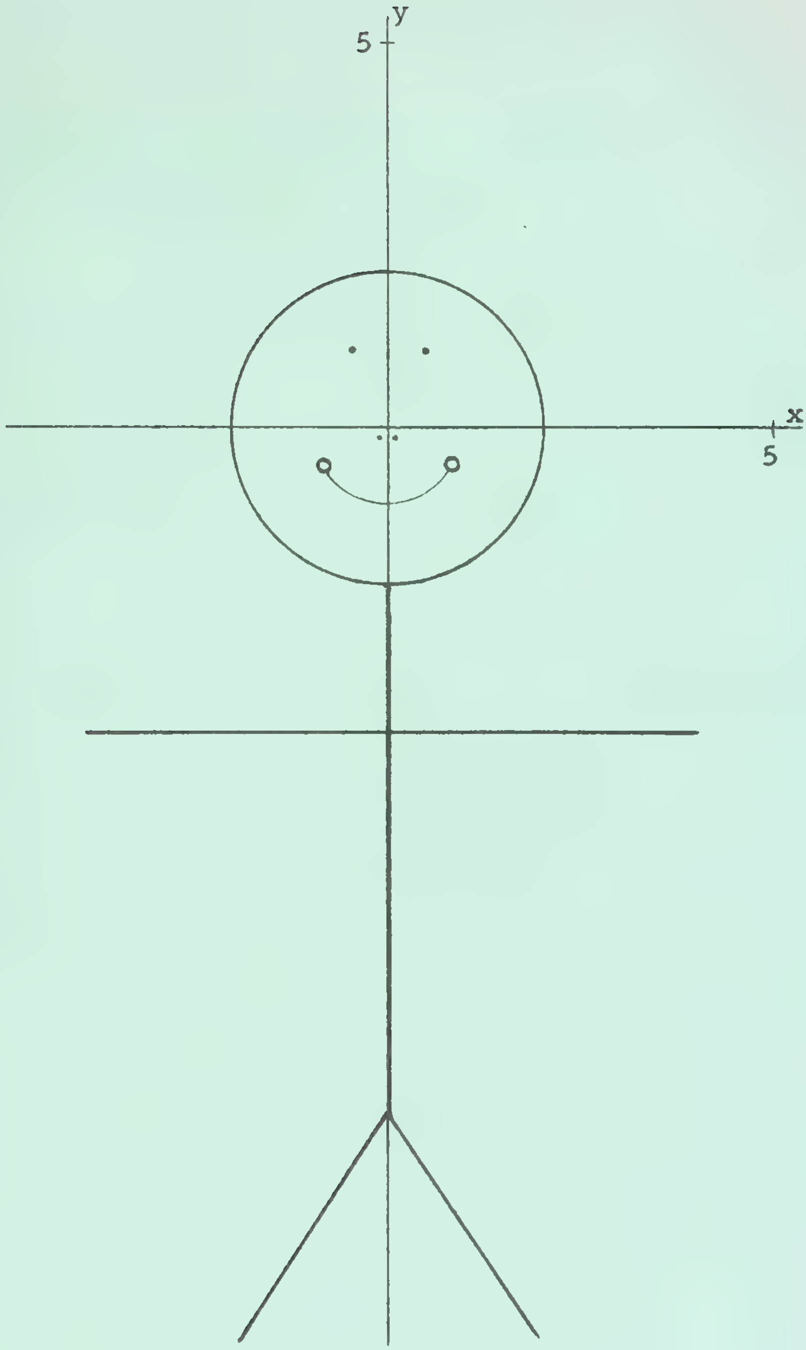
14.



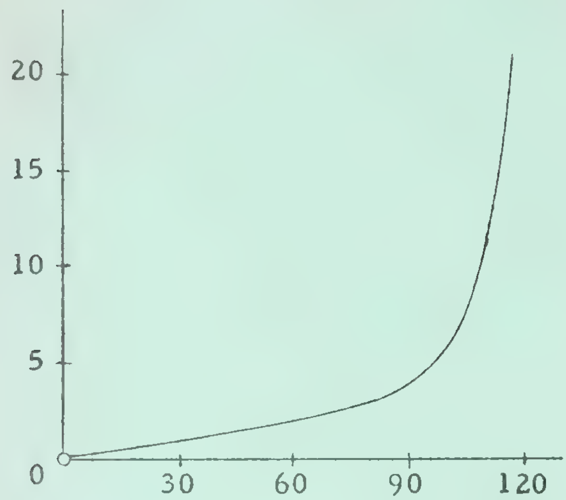
☆15.



☆16.



D. 1.



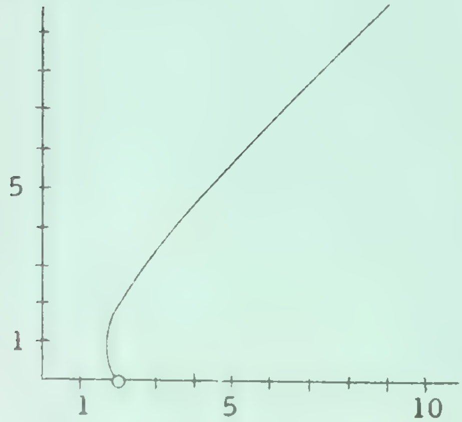
[Function is
$\{(x, y),\ 0 < x < 120:$
$y = \frac{2 \sin x°}{\sin(120 - x)°}\}.]$

2. (a) 1 (b) 2 (c) 4
(d) 0; 120

3. (a) The inch-measure of $\overleftrightarrow{AC}$ must be approximately 1.31; that of $\overleftrightarrow{BC}$, 1.76. So, the graph consists of a single dot.

(b) The inch-measure of $\overleftrightarrow{BC}$ must be approximately 2.64, the degree-measure of $\angle B$ is approximately 79. The relation consists of a single ordered pair.

4.



(a) one

(b) two

(c) none

[From the Law of Cosines we find that the relation is $\{(x, y),\ y > 0:\ x^2 - (y - 1)^2 = 3\}.]$

Correction. Part E, Exercise 2(h):

$H = \{(x, y) \in I^+ \times I^+,\ x > 1 : x < 30 \ldots$

E. 1. [In listing solutions we use, here, 'R' for the set of real numbers.]

1. R, R
2. R, R
3. $\{x: -5 \leq x \leq 5\}$, $\{x: -5 \leq x \leq 5\}$
4. $\{x: -10 \leq x \leq 10\}$, $\{x: -10 \leq x \leq 10\}$
5. R, $\{8\}$
6. $\{8\}$, R
7. R, R
8. $\{x: x < 2\}$, $\{x: x > 3\}$
9. $\{x: x \neq 0\}$, $\{x: x \neq -1\}$
10. R, R
11. $\{x: -5 \neq x \neq 5\}$, R
12. R, R
13. R, $\{3, -1\}$
14. $\{x: 1 \leq x \leq 5\}$, $\{x: 2 \leq x \leq 6\}$
15. R, R
16. $\{x: x \neq 0\}$, $\{x: x \neq 0\}$
17. $\emptyset$, $\emptyset$
18. $\{x: x \leq 2 \text{ or } x \geq 3\}$, R
19. R, R
20. R, $\{x: x \geq -\frac{9}{4}\}$

2. (a) $\{x \in I^+: x \neq 1\}$, I^+, I^+

(b) $\{x \in I: x \text{ is odd}\}$, $\{x \in I: x \text{ is odd}\}$, $\{x \in I: x \text{ is odd}\}$

(c) I, I, I

(d) $\{x \in I: x \text{ is odd}\}$, $\{x \in I: x \text{ is even}\}$, I

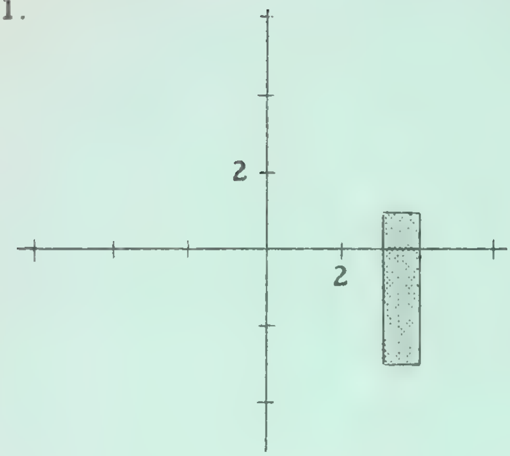
(e) I^+, the set of squares of positive integers, I^+

(f) I, $\{13\}$, I (g) $\emptyset$, $\emptyset$, $\emptyset$

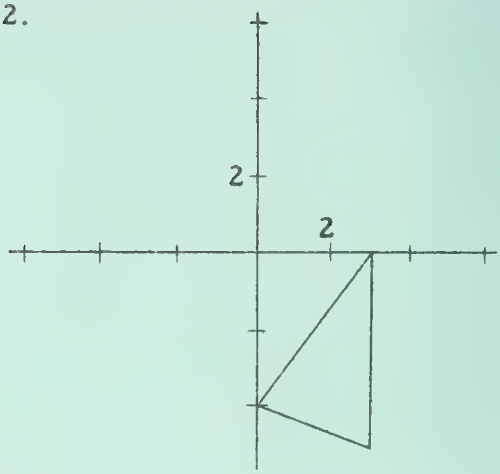
(h) $\{x \in I^+: 1 < x < 30\}$,
$\{1, 3, 4, 6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 21, 22, 28, 36\}$,
$\{x \in I^+: x < 30 \text{ or } x = 36\}$

3. (a) Yes (b) No (c) No

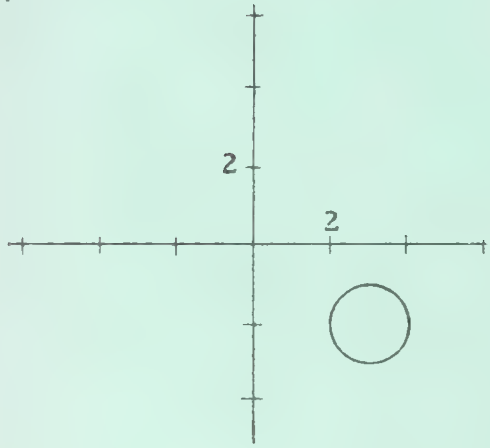
F. 1.



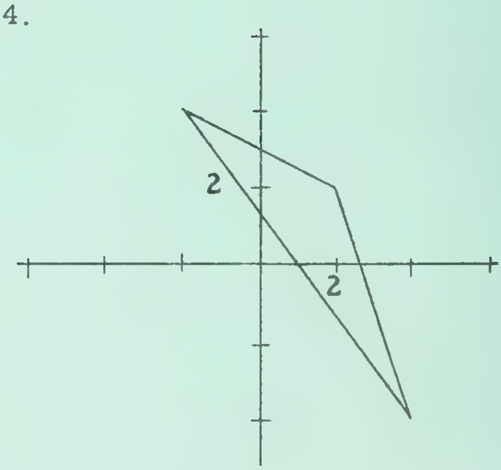
2.



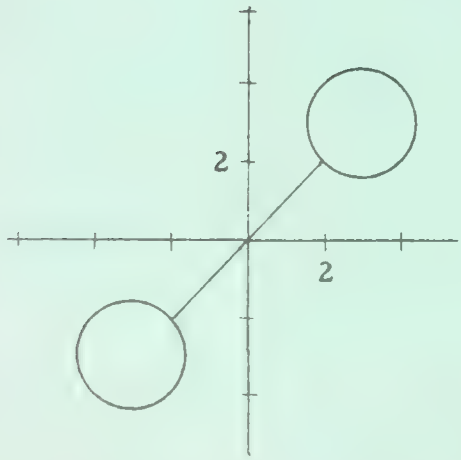
3.



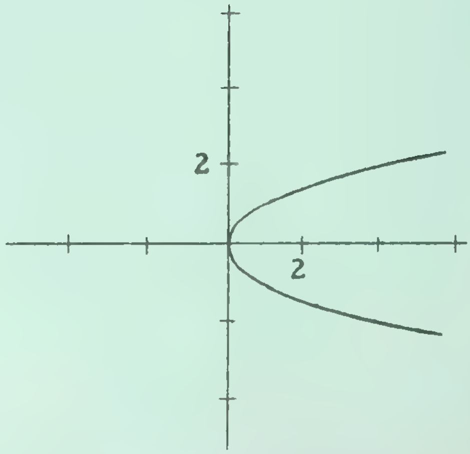
4.



5.



6.



G. 1. The relations in Exercises 7, 15, 17, and 19 are reflexive; those in Exercises 3, 4, 15, 16, and 17 are symmetric.

2. The relations in parts (b), (d), (e), and (f) are reflexive. Those in parts (b), (c), (d), and (e) are symmetric.

3. (1, 1), (3, 3), (5, 5), (7, 7), (9, 9), (11, 11), (13, 13)

4. The relations in parts (a), (b), and (e) are reflexive.

5. (a) $\{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)\}$

   (b) $\{1, 2, 3, 4, 5\} \times \{1, 2, 3, 4, 5\}$

6. All the relations listed are symmetric. [Once one has seen that $\widetilde{S}_1$, $S_1 \cup S_2$ and $S_1 \cap S_2$ must be symmetric if S_1 and S_2 are, the remaining parts are trivial. Take part (i), for example. Since S_1 and S_2 are symmetric, their complements are symmetric. Since $\widetilde{S}_1$ and $\widetilde{S}_2$ are symmetric, their union is symmetric. Since $\widetilde{S}_1 \cup \widetilde{S}_2$ is symmetric, so is its complement.]

7. The relations in parts (c), (e), (f), (g), and (h) are symmetric.

8. (a) The set of symmetric relations among the members of a set S is closed under unioning, intersecting, and complementing.

   (b) The set of reflexive relations among the members of a set S is closed under unioning and intersecting [but not under complementing].

> Corrections. [page 5-247]
> In the first line of Exercise 4, and in the second line of Exercise 5, change 'Exercise 8' to 'Exercise 7'.
> In line 16, underline 'antisymmetric relations'.
> In Exercise 8, left column of exercise letters, change '(b)' to '(c)' and '(c)' to '(e)'.

☆ H. 1. reflexive: (b), (e), (i), (j); irreflexive: (a), (h), (j); neither: (c), (d), (f), (g)

2. reflexive: (b), (c), (e), (f), (h), (j);
irreflexive: (a), (d), (g), (i) [For some values of 'R' and 'N', the relation named in part (a), that in part (d), or that in part (i) is $\emptyset$. In such a case, the relation in question is reflexive as well as irreflexive.]

3. (a) (iii) (b) (i) T (ii) F (iii) F

(c) (i) There is an $(x, y) \in P \times P$ such that $y\ F\ x$ and $x \not F\ y$.

(ii) There is an $(x, y) \in P \times P$ such that $y\ B\ x$ and $x \not B\ y$.

(iii) There is an $(x, y) \in P \times P$ such that $y\ M\ x$ and $x \not M\ y$.

4. (a), (e)

5. (a) [There are many correct answers; for example, the relations $<$, and being a grandchild of, are two such.]

(b) [Any relation R among the members of a set S such that (1) there is an $(x, y) \in S \times S$ such that $(x, y) \in R$ and $(y, x) \notin R$ and (2) there is an $(x, y) \in S \times S$ such that $(x, y) \in R$ and $(y, x) \in R$.]

(c) [Any relation R among the members of a set S such that (1) there is no $(x, y) \in S \times S$ such that $(x, y) \in R$ and $(y, x) \in R$ and (2) there is no $(x, y) \in S \times S$ such that $(x, y) \in R$ and $(y, x) \notin R$--but, the only such relation R is $\emptyset$.]

6. (a)

7. (b), (c), (e)

8. (a) (1, 1), (2, 2), (3, 3), (4, 4), (5, 5)

(b) (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 3) [Or, instead of (1, 3), any ordered pair not previously listed.]

(c) (1, 3), (3, 1), (2, 4), (4, 2), (1, 5), (5, 1) [There are other correct answers.]

(d) (1, 3), (2, 4), (1, 5) [There are other correct answers.]

(e) (1, 3), (2, 4), (1, 5) [There are other correct answers.]

(f) [Impossible. An asymmetric relation is irreflexive, and an irreflexive relation is reflexive only if its field is $\emptyset$.]

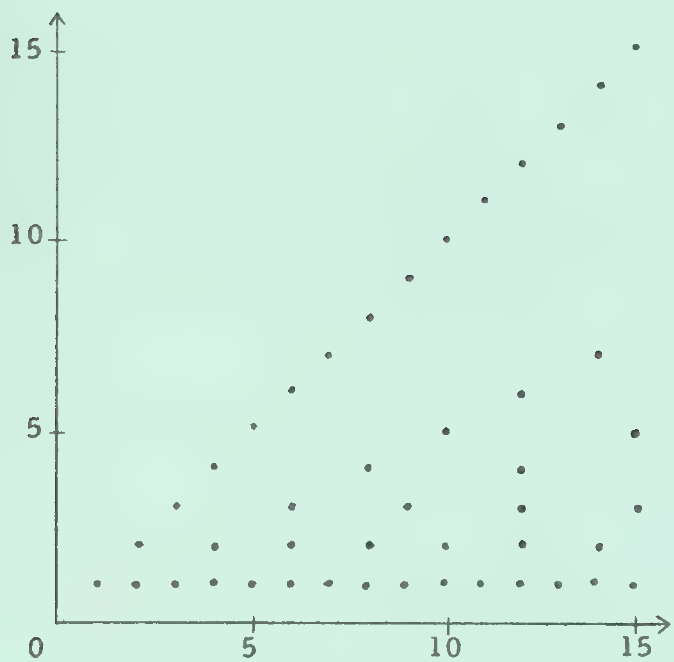
I. 1. (a) The relation is neither reflexive nor symmetric.

(b) The domain of the relation is {Art, Cal, Eli}; its range is C.

(c) Include 5 additional ordered pairs to obtain a reflexive relation; 6 to obtain a symmetric relation.

2. (a) Eli (b) Art (c) Dot; Bob

3. (a)



$\mathcal{S}_{|} = I^{+} = \mathcal{R}_{|}$

(b) $|$ is reflexive, but not symmetric [it is antisymmetric].

☆J.

Exercise	reflexive	irreflexive	symmetric	asymmetric	antisymmetric	transitive	intransitive
1.		✓					
2.	✓		✓			✓	
3.	✓		✓				
4.		✓	✓				
5.	✓		✓			✓	
6.		✓		✓	✓		✓
7.	✓		✓			✓	
8.	✓		✓			✓	
9.	✓		✓			✓	
10.	✓		✓			✓	
11.	✓		✓			✓	
12.	✓				✓	✓	
13.	✓	✓	✓	✓	✓	✓	✓
14.	✓		✓		✓	✓	
15.		✓		✓	✓	✓	
16.	✓				✓	✓	
17.			✓				
18.		✓	✓				✓
19.		✓		✓	✓	✓	
20.		✓		✓	✓		✓
21.			✓				
22.	✓					✓	
23.		✓	✓				
* 24.		✓	✓				✓
25.	✓		✓			✓	

* The set named is a relation among geometric figures in a plane.

☆26. (a) Suppose that $a \in \mathfrak{D}_R$. Since R is symmetric, $a \in \mathfrak{R}_R$. Hence, there is a $b \in \mathfrak{R}_R$ such that $(a, b) \in R$. Since R is symmetric, $(b, a) \in R$. Since R is transitive and since $(a, b) \in R$ and $(b, a) \in R$, $(a, a) \in R$. So, R is reflexive.

(b) Suppose that $a \in \mathfrak{D}_R$. Since R is asymmetric, not both (a, a) and (a, a) belong to R--that is, $(a, a) \notin R$. So, R is irreflexive.

(c) Each of Exercises 1, 4, 18, 23, and 24 gives a counter-example--that is, a relation which is irreflexive but not asymmetric.

(d) If R is transitive and asymmetric then, by part (b), R is irreflexive. On the other hand, if R is transitive and irreflexive then, if $(a, b) \in R$, $(b, a) \notin R$ [since $(a, a) \notin R$]. So, a transitive and irreflexive relation is asymmetric.

(e) If R is reflexive then, for each $a \in \mathfrak{D}_R$, $(a, a) \in R$. Hence, for each $a \in \mathfrak{D}_R$, $(a, a) \notin \widetilde{R}$. So, $\widetilde{R}$ is irreflexive.

(f) If $(a, b) \in \widetilde{R}$ then $(a, b) \notin R$ and, if R is symmetric, $(b, a) \notin R$--that is, $(b, a) \in \widetilde{R}$. Hence, $\widetilde{R}$ is symmetric.

(g) Each of Exercises 2, 5, 9, 10, 11, 12, 14, 19, 22, and 25 gives a counter-example.

(h) If R is asymmetric then, if $(a, b) \in R$, $(b, a) \notin R$. Hence, in particular, if $a \neq b$ then not both (a, b) and (b, a) belong to R. So, R is antisymmetric.

(i) Each of Exercises 12, 14, and 16 gives a counter-example.

Correction.	On page 5-252, line 1: ...the function in Exercise 3(c) whose...

K. 1. (a), (c), and (f) are functions, the others are not.

2. The converse of (c) is a function, those of the others are not.

3. (a) Yes (b) Yes (c) Yes (d) yes

(e) No [r_2 and r_5 have the same perimeter but different area-measures.]; No [r_3 and r_4 have the same area-measure but different perimeters.]

(f) (1) 12 (2) 10 (3) 24 (4) $\frac{13}{2}$

4. (a) (1) 5 (2) 2 (3) 3 (4) 3

(5) 6 (6) 1 (7) 7 (8) 12

(9) {1, 2, 3, 4, 5, 6} (10) {1, 2, 3, 5, 6}

(b) (1) −2 (2) 28 (3) −12 (4) −27

(5) 8 (6) −17 (7) 28 (8) 3

(9) 5 (10) $\frac{13}{5}$ (11) 10a - 12 (12) 15b - 7

(13) c - 12 (14) d

(c) (1) 10 (2) 16 (3) 25 (4) 37

(5) 91 (6) −65

(7) (i) 4 (ii) 8 (iii) $\frac{13}{3}$ (iv) 1 (v) 1

(vi) $-\frac{1}{5}$ (vii) −1 (viii) The solution set is the set of real numbers.

(d) (1) 2 (2) $\frac{3}{5}$ (3) $\frac{2}{7}$ (4) 3 (5) $\frac{2 - 6a}{7}$

(6) $\frac{3 - 18k}{5}$ (7) $\frac{2 - 42m}{3}$ (8) $\frac{13 - 35n}{6}$ (9) $\frac{11}{27}$

(e) (1) -4 (2) 36 (3) 14 (4) 0 (5) 0

(6) $-\frac{25}{4}$ (7) $4a^2 - 18a + 14$ (8) $25b^2 - 45b + 14$

(9) $4m^2 - 22m + 24$ (10) $9n^2 - 15n$ (11) $p^2 - \frac{25}{4}$ (12) $4k^4 - 18k^2 + 14$

(f) (1) 17 (2) -3 (3) 4 (4) $-\frac{5}{4}$ (5) $4k + 5$

(6) $4(k + h) + 5$ (7) $4b$ (8) $4d$ (9) $-\frac{5}{3}$

(g) (1) 0 (2) 0 (3) 4 (4) 3 (5) 3

(6) $[-\frac{1}{4} \notin \mathscr{D}_s]$ (7) $[\frac{5}{2} \notin \mathscr{D}_s]$ (8) $\frac{96}{25}$ (9) 4

☆(10) If $s(t_0) = s(t_1)$ then

$$16t_0 - 16t_0^2 = 16t_1 - 16t_1^2.$$

So,

$$16t_0 - 16t_1 = 16t_0^2 - 16t_1^2,$$

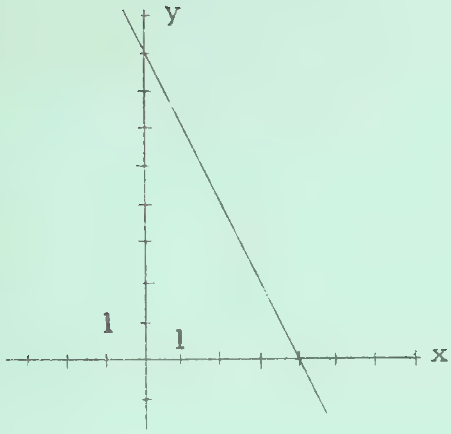
$$16(t_0 - t_1) = 16(t_0^2 - t_1^2),$$

$$16(t_0 - t_1) = 16(t_0 - t_1)(t_0 + t_1),$$

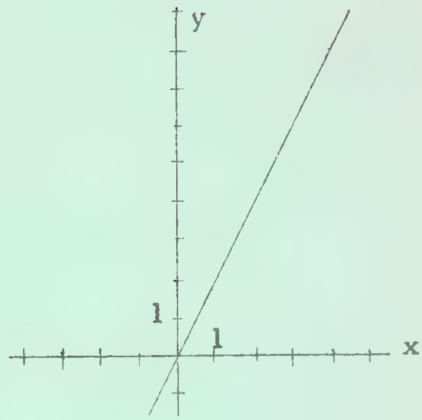
and

$$1 = t_0 + t_1. \quad [t_0 - t_1 \neq 0]$$

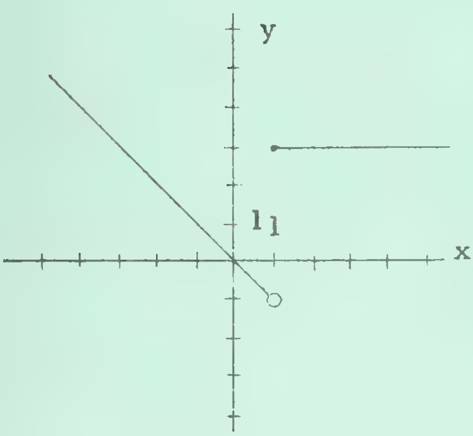
L. 1.



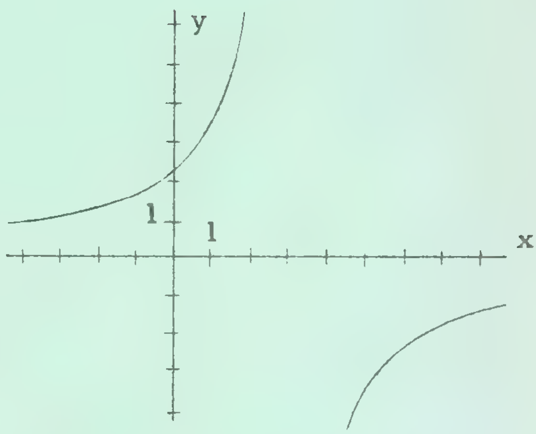
2.



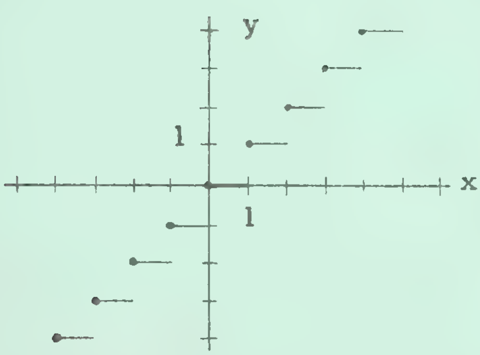
3.



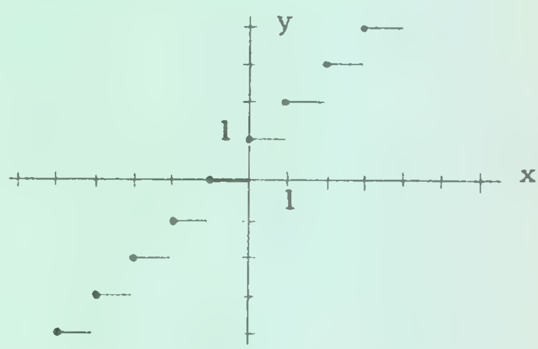
4.



5.



6.

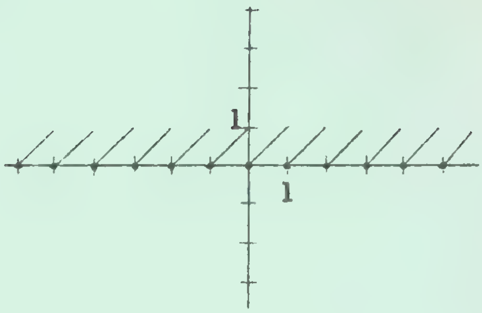


[Note the colloquial way of referring to this function.]

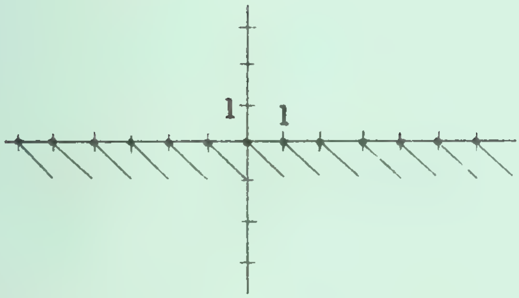
7.

[Same as Exercise 6.]

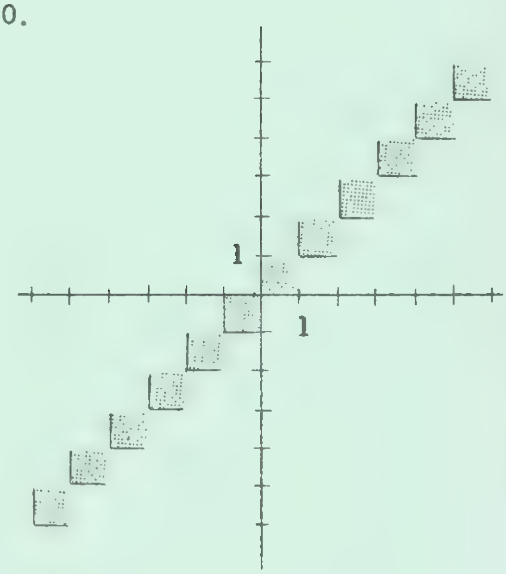
8.



9.

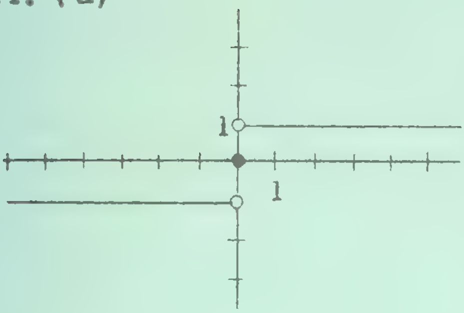


10.

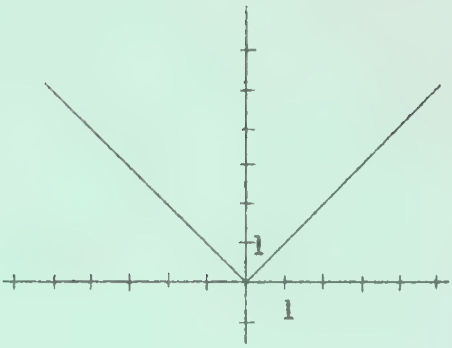


[Not a function]

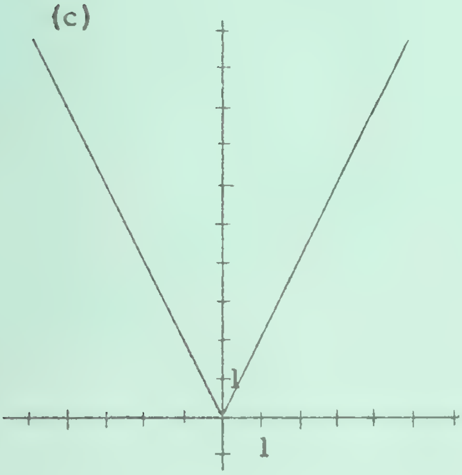
11. (a)



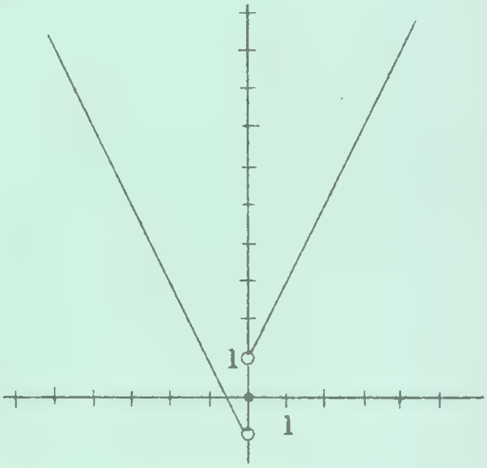
(b)



(c)



(d)



M. 1. (1, 10), (2, 11), (3, 12), (4, 12), (5, 13), (6, 13)

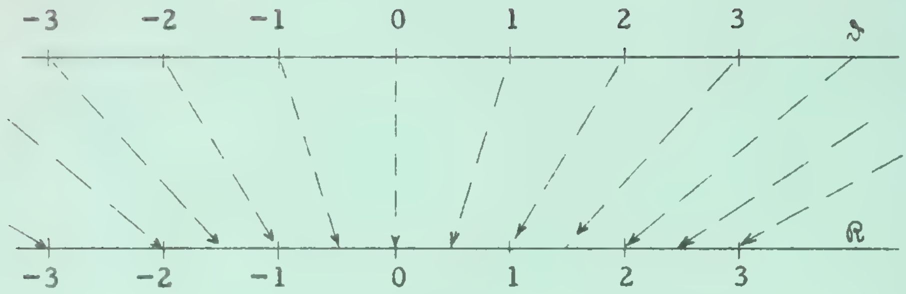
2. $\{(x, y) : y = -x\}$

3. $16; -\frac{13}{2}$

4. $\{(x, y) : y = 2x^2 + 1\}$

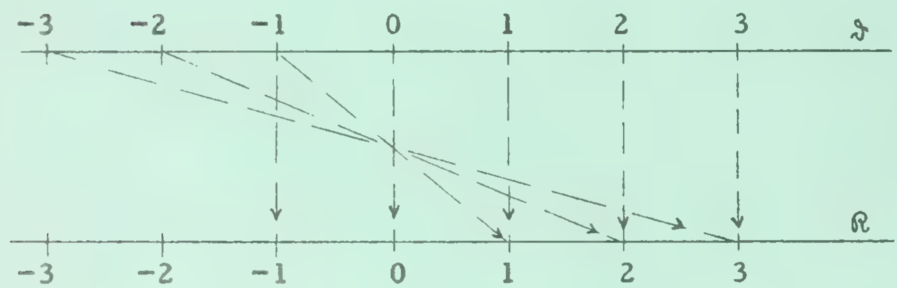
5. $\{(x, y) : y = 3x + 7\}$
[Ex. 5 is on page 5-255.]

7. (a)



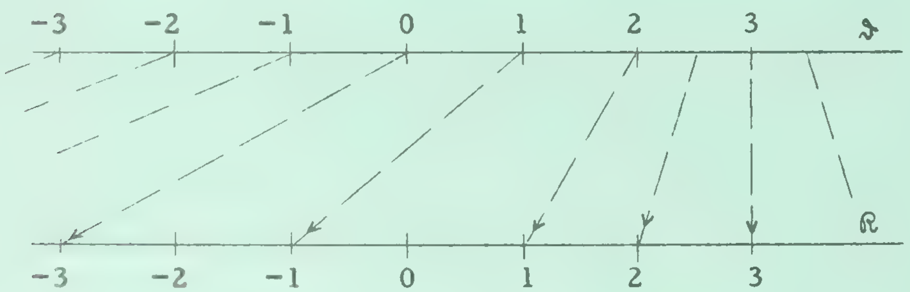
-3 -2 -1 0 1 2 3
$\mathcal{J}$
$\mathcal{R}$
-3 -2 -1 0 1 2 3

(b)



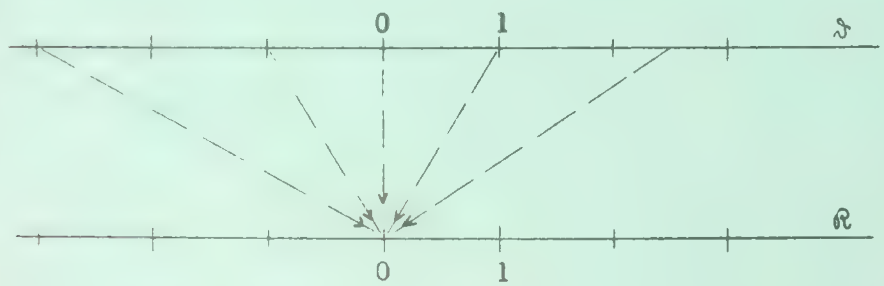
-3 -2 -1 0 1 2 3
$\mathcal{J}$
$\mathcal{R}$
-3 -2 -1 0 1 2 3

(c)



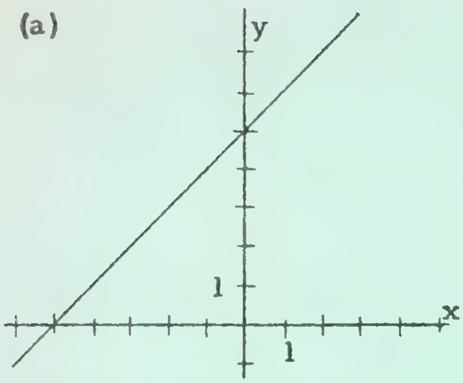
-3 -2 -1 0 1 2 3
$\mathcal{J}$
$\mathcal{R}$
-3 -2 -1 0 1 2 3

(d)

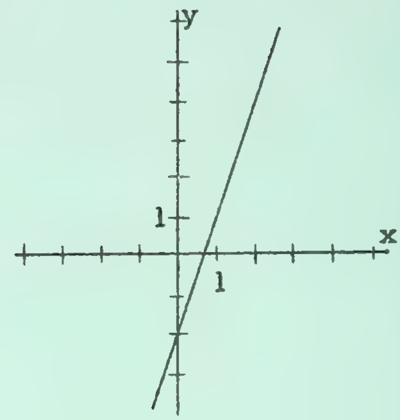


0 1
$\mathcal{J}$
$\mathcal{R}$
0 1

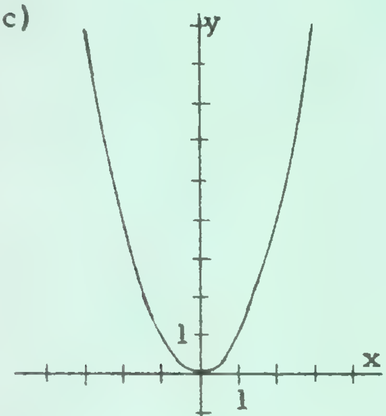
6. (a)



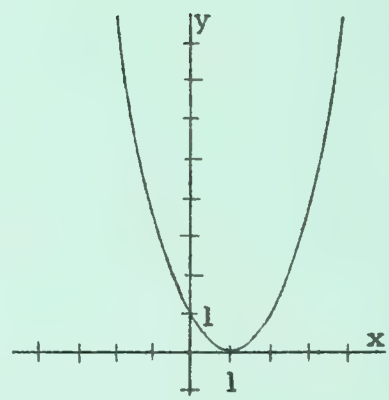
(b)



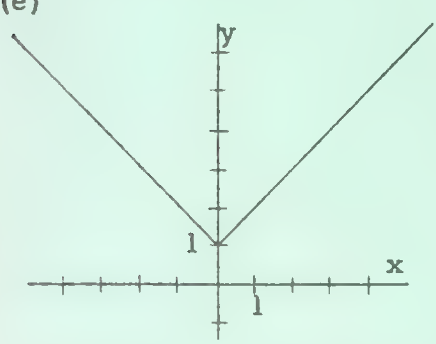
(c)



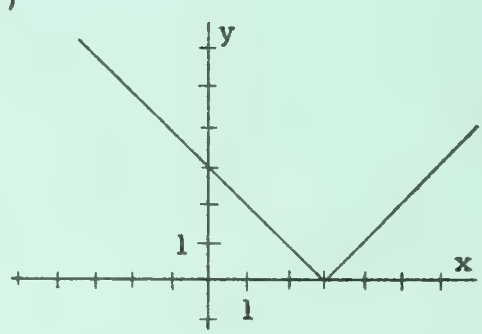
(d)



(e)

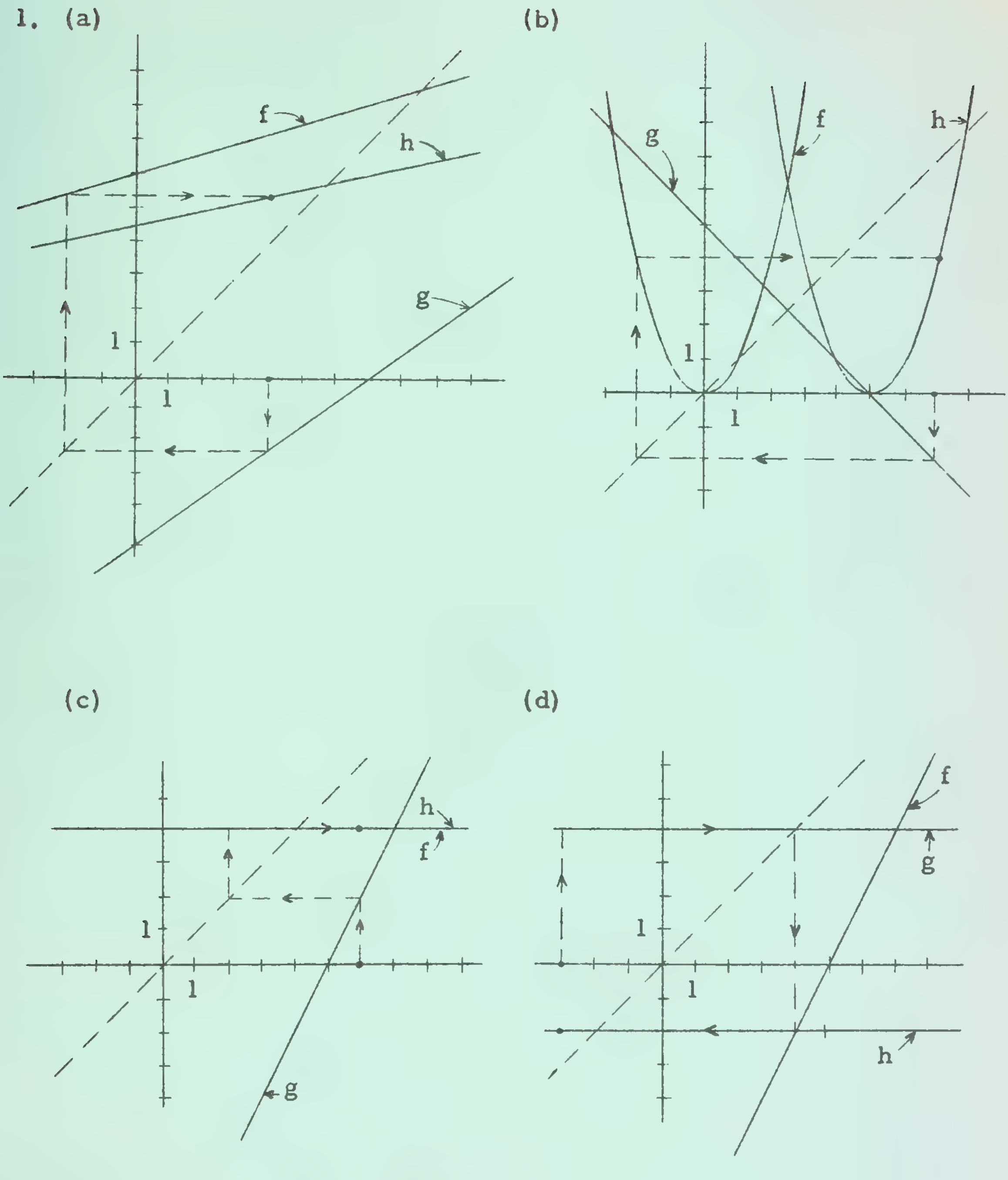


(f)



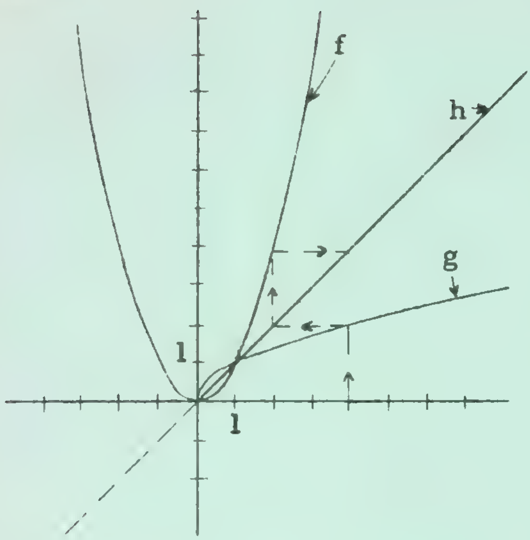
[On comparing parts (c) and (d) of Exercise 6, and parts (e) and (f), students may discover if, for some function f and some number k, a function g is defined by: $g(x) = f(x - k)$, then the graph of g is the graph of f shifted k units to the right; and that if a function h is defined by: $h(x) = f(x) + k$, then the graph of h is the graph of f shifted k units up.]

N. 1.

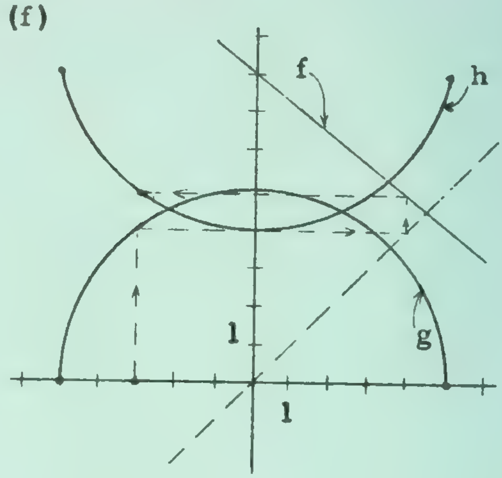


Correction.	In the diagram for (e), mark the "vertical parabola" with an 'f' and the "horizontal" one with a 'g'.

(e)



(f)



2. $x \to 6x + 14$

   (a) $x \to 7x - 4$ (b) $x \to 4x - 9$ (c) $x \to 3 - 5x$

   (d) $x \to x$ (e) $x \to (8x - 4)^2$

   (f) $x \to 3(2x - 5)^2 - 2(2x - 5) + 5$ [or: $x \to 12x^2 - 64x + 90$]

<u>O.</u> 1. (a) $\{(x, y) : y = x + 2\}$ (b) $\{(x, y) : y = x7\}$

   (c) $\{(x, y) : y = xx\}$ (d) $\{(x, y) : x = y + 2\}$

   (e) $\{(x, y) : x = y7\}$

   (f) $\{(x, y), y \in N: y = |x|\}$, or: $\{(x, y), y \in N: x = {}^{+}y \text{ or } x = {}^{-}y\}$

   (g) $\{(x, y) : x + y = 0\}$ (h) $\{(x, y) : y = x\}$

   (i) $\{(x, y) : y + x = 0\}$ (j) $\{(x, y) : y = x2\}$

   (k) $\{(x, y) : x = y3)$

2. The functions named in parts (a), (c), (d), (e), (g), (h), (i), and (j) have inverses; that is, functions A, C, D, f, F, G, H, and K have inverses. [Functions B, g, and M do not have inverses because $B(5) = B(11)$, $g(1) = g(-1)$, and $M(0) = 12 = M(1)$.]

$A^{-1} = \{(7, 4), (9, 8), (11, 12), (13, 16)\}$

$C^{-1}(x) = x + 1$, $\mathcal{D}_{C^{-1}}$ = the set of real numbers

$D^{-1}(x) = \frac{x - 5}{3}$, $\mathcal{D}_{D^{-1}}$ = the set of real numbers

$f^{-1}(x) = \frac{x + 5}{4}$, $\mathcal{D}_{f^{-1}}$ = the set of real numbers

$F^{-1} = \{(x, y): y = \frac{x + 3}{3}\}$ $\qquad$ $G^{-1} = \{(x, y): y = \frac{4 - x}{9}\}$

$H^{-1} = \{(x, y): 2x + y = 8\}$ $\qquad$ $K^{-1} = \{(x, y): 5x - 3y + 7 = 0\}$

3. (a) 2 (b) 2 (c) 9 (d) 16 (e) $\frac{2}{3}$

(f) 2 (g) 14 (h) 14 (i) $\frac{15a + 2}{3}$ (j) $7b + 2$

(k) (i) -1 (ii) $-\frac{4}{5}$ (iii) -1 (iv) $-\frac{4}{5}$ (v) $-\frac{6}{11}$

(vi) $-\frac{8}{11}$ (vii) $-\frac{26}{29}$ (viii) $-\frac{24}{29}$ (ix) $-\frac{24}{29}$ (x) $-\frac{26}{29}$

[Note that it is not a coincidence that, in part (k), (i) and (iii) have the same answer, as do (ii) and (iv), (vii) and (x), and (viii) and (ix). In each case, the two equations are equivalent by virtue of the fact that, for each function h which has an inverse,

$$\forall_{x \in \mathcal{D}_h} \; h^{-1}(h(x)) = x$$

and

$$\forall_{x \in \mathcal{R}_h} \; h(h^{-1}(x)) = x.$$

So, for example, if $a \in \mathcal{D}_f$ then $f^{-1}(f(a)) = a$; and if, in particular, $f(a) = a$ then $f^{-1}(f(a)) = f^{-1}(a)$. Hence, if $f(a) = a$ then $f^{-1}(a) = a$. Similarly, if $f^{-1}(a) = a$ then $f(a) = a$. Consequently, (i) and (iii) are equivalent.]

4. (b) $x \rightarrow \frac{x - 10}{5}$ (c) $x \rightarrow \frac{x + 12}{2}$ (d) [no inverse]

(e) $x \rightarrow \frac{x + 5}{7}$ (f) [no inverse] (g) $x \rightarrow \frac{x - 4}{3}$

5. (a) 3 (b) $-\frac{5}{2}$ (c) 12 (d) [none]

(e) $\frac{5}{6}$ (f) $-\frac{1}{2}$ (g) -2

P. 1. given set = $\{(2, 1), (3, 3), (4, 5)\}$;
image = $\{(5, -4), (6, -2), (7, 0)\}$

2. given set = $\overline{(2, 1), (4, 5)}$; image = $\overline{(5, -4), (7, 0)}$

3. given set = $\{(4, 4), (5, 3), (6, 2), (7, 1)\}$;
image = $\{(4, -4), (5, -3), (6, -2), (7, -1)\}$

4. given set = $\overline{(0, 1), (3, 7)}$; image = $\overline{(0, -1), (3, -7)}$

5. given set = $\{(x, y) : y = 2x + 1\}$; image = $\{(x, y) : y = -2x - 1\}$

6. given set = $\{(x, y) : y = x\}$; image = $\{(x, y) : y = -x\}$

7. given set = $\overline{(3, 0), (3, 4)}$; image = $\overline{(0, 3), (-4, 3)}$

8. given set = $\overline{(0, 3), (3, 3)} \cup \overline{(3, 3), (3, 0)}$;
image = $\overline{(-3, 0), (-3, 3)} \cup \overline{(-3, 3), (0, 3)}$

9. given set = $\overline{(-3, 0), (-3, 3)} \cup \overline{(-3, 3), (0, 3)}$
image = $\overline{(0, 3), (-3, -3)} \cup \overline{(-3, -3), (-3, 0)}$

10. given set = image 11. [on page 5-261] S

12. S   13. S   14. S

[Of course, the answers for Exercises 13 and 14 are clear once one has answered Exercises 11 and 12.]

15. the rectangle whose vertices are (−2, 3), (−2, −3), (2, −3), and (2, 3)

16. (a) T   (b) R   (c) R   (d) R   (e) R

17. No. [Each point other than the origin has different images under $f \circ g$ and $g \circ f$. In fact, $f \circ g: (x, y) \to (-x, y)$, and $g \circ f: (x, y) \to (x, -y)$.]

*

To reason out how many ways the cardboard square can be placed on the drawing, note that the corner A can be placed in any one of four positions, after which the corner B, since it is adjacent to A, must be placed at one of the two corners of the drawing adjacent to that chosen for A. Now D must be placed at the other of these two corners, and C at the one remaining corner. So, there are just $4 \cdot 2$ ways of placing this square on the drawing.

*

18. The five remaining rows in the table will be: BADC, CBAD, CDAB, DCBA, and DABC. [These correspond, respectively, to f_8, f_7, f_3, f_6, and f_4 in the list at the foot of page 5-263.]

*

Answers to questions at the bottom of page 5-263.

f_4 is a counter-clockwise rotation through three quarter-turns; so, $f_4 = f \circ f \circ f$.

$f_8 = g \circ f_4 = g \circ f \circ f \circ f$ [Note that, since composition of functions is associative, brackets are unnecessary.]

> **Correction.** On page 5-264, in Exercise 21, change '$(f_2 \circ f_6)^{-1}$' to '$[f_2 \circ f_6]^{-1}$'.

19.

$\circ$	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
f_1	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
f_2	f_2	f_3	f_4	f_1	f_8	f_5	f_6	f_7
f_3	f_3	f_4	f_1	f_2	f_7	f_8	f_5	f_6
f_4	f_4	f_1	f_2	f_3	f_6	f_7	f_8	f_5
f_5	f_5	f_6	f_7	f_8	f_1	f_2	f_3	f_4
f_6	f_6	f_7	f_8	f_5	f_4	f_1	f_2	f_3
f_7	f_7	f_8	f_5	f_6	f_3	f_4	f_1	f_2
f_8	f_8	f_5	f_6	f_7	f_2	f_3	f_4	f_1

$(x, y) \to (x, y)$	identity
$(x, y) \to (-y, x)$	f
$(x, y) \to (-x, -y)$	$f \circ f$
$(x, y) \to (y, -x)$	$f \circ f \circ f$
$(x, y) \to (y, x)$	g
$(x, y) \to (x, -y)$	$g \circ f$
$(x, y) \to (-y, -x)$	$g \circ f \circ f$
$(x, y) \to (-x, y)$	$g \circ f \circ f \circ f$

The work of completing the composition table can be carried out in several ways. For example, one may fill out the f_4-row by noticing that f_4 first replaces the first component of a point by its opposite and then interchanges the components. Carrying out this operation on the results of applying f_1, f_2, ..., f_8 to (x, y) shows that

$$[f_4 \circ f_1]((x, y)) = f_4((x, y)) = (y, -x),$$

$$[f_4 \circ f_2]((x, y)) = f_4((-y, x)) = (x, y),$$

$$[f_4 \circ f_3]((x, y)) = f_4((-x, -y)) = (-y, x),$$

$$[f_4 \circ f_4]((x, y)) = f_4((y, -x)) = (-x, -y),$$

$$[f_4 \circ f_5]((x, y)) = f_4((y, x)) = (x, -y),$$

$$[f_4 \circ f_6]((x, y)) = f_4((x, -y)) = (-y, -x),$$

$$[f_4 \circ f_7]((x, y)) = f_4((-y, -x)) = (-x, y),$$

$$[f_4 \circ f_8]((x, y)) = f_4((-x, y)) = (y, x).$$

So, $f_4 \circ f_1 = f_4$, $f_4 \circ f_2 = f_8$, $f_4 \circ f_3 = f_2$, etc. The other rows can be filled out in a similar manner.

Here is a more interesting way to do it. As noted at the bottom of page 5-263, $f_2 = f$, $f_3 = f \circ f$, $f_4 = f \circ f \circ f$, $f_5 = g$, $f_6 = g \circ f$, $f_7 = g \circ f \circ f$, and $f_8 = g \circ f \circ f \circ f$. If we denote f_1 by 'i' [for 'identity mapping'] then $f \circ f \circ f \circ f = i = g \circ g$. Also, $f \circ i = f = i \circ f$, and $g \circ i = g = i \circ g$. Now,

$$f_4 \circ f_1 = [f \circ f \circ f] \circ i = [f \circ f] \circ [f \circ i] = f \circ f \circ f = f_4,$$

$$f_4 \circ f_2 = [f \circ f \circ f] \circ f = i = f_1,$$

$$f_4 \circ f_3 = [f \circ f \circ f] \circ [f \circ f] = f \circ [f \circ f \circ f \circ f] = f \circ i = f = f_2,$$

$$f_4 \circ f_4 = f_4 \circ [f \circ f \circ f] = f_4 \circ [f_3 \circ f] = [f_4 \circ f_3] \circ f = f_2 \circ f = f_3,$$

$$f_4 \circ f_5 = [f \circ f \circ f] \circ g = ?$$

To answer this last question, note that $[f \circ g]((x, y)) = f((y, x)) = (-x, y) = f_8((x, y))$. So, $f \circ g = f_8 = g \circ f \circ f \circ f = g \circ f_4$. So,

$$f_4 \circ f_5 = f \circ f \circ f \circ g = f \circ f \circ [f \circ g] = f \circ f \circ [g \circ f_4] = f \circ [f \circ g] \circ f_4 = f \circ [g \circ f_4] \circ f_4$$

$$= [f \circ g] \circ [f_4 \circ f_4] = [g \circ f_4] \circ [f_4 \circ f_4] = [g \circ f_4] \circ f_3 = g \circ [f_4 \circ f_3]$$

$$= g \circ f_2 = g \circ f = f_6,$$

$$f_4 \circ f_6 = f_4 \circ [f_5 \circ f] = [f_4 \circ f_5] \circ f = f_6 \circ f = f_7,$$

$$f_4 \circ f_7 = f_4 \circ [f_6 \circ f] = [f_4 \circ f_6] \circ f = f_7 \circ f = f_8,$$

$$f_4 \circ f_8 = f_4 \circ [f_7 \circ f] = [f_4 \circ f_7] \circ f = f_8 \circ f = g \circ f \circ f \circ f \circ f = g \circ i = g = f_5.$$

20. f_2, f_3, and f_4 are rotations; f_5, f_6, f_7, and f_8 are reflections.

21. $f_2^{-1} = f_4$; $f_6^{-1} = f_6$; $[f_2^{-1} \circ f_6]^{-1} = f_5^{-1} = f_5$; $f_6^{-1} \circ f_2^{-1} = f_6 \circ f_4 = f_5$

*

The composition table of Exercise 19 gives a good deal of information about the behavior of the functions f_1 through f_8 with respect to the operation of function composition. To see better what information is contained in the table, let's forget how we constructed it and suppose merely that someone has told us that there is a set G of eight things $\{f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8\}$ and an operation $\circ$ on them, and that the table defines this operation. That is, if s and t are any members of G then $s \circ t$ is the member of G listed in the s-row and t-column of the table.

One thing that we could learn from the table is that $\circ$ is associative. We could learn this by checking each of 8^3 instances of the associative principle. [For example, according to the table, $[f_4 \circ f_3] \circ f_5 = f_2 \circ f_5 = f_8$, and $f_4 \circ [f_3 \circ f_5] = f_4 \circ f_7 = f_8$. So, $[f_4 \circ f_3] \circ f_5 = f_4 \circ [f_3 \circ f_5]$.]

Another thing we might notice is that each member of G is listed in each row of the table. What this means is that if s and u are members of G then there is a $t \in G$ such that $s \circ t = u$. For, given s and u, we can find u listed somewhere in the s-row. So, we can solve equations like '$f_5 \circ x = f_4$'. [According to the table, the root of this equation is f_8.] Similarly, since each of the eight things is listed in each column, we can solve equations like '$x \circ f_5 = f_4$'. [According to the table, the root of this equation is f_6.]

So, the table tells us that $\circ$ is an associative operation on the members of G, and that equations of the forms '$s \circ x = u$' and '$x \circ s = u$' have roots. More briefly put, what the table tells us is that G is a group with respect to the operation $\circ$

You are already acquainted with many groups. For example, the integers form a group with respect to addition; so do the even integers, the rational numbers, and the real numbers. Also, the positive rational numbers form a group with respect to multiplication; so does the set of all nonzero rational numbers. So do the set of positive real numbers, and the set of all nonzero real numbers. Notice that each of these groups has an additional property which G does not have. The group operation $\circ$ is not commutative, while, as you know, addition and multiplication of real numbers are commutative. Groups for which the group operation is commutative are called commutative groups. Although G is not commutative it has several commutative subgroups--that is, subsets which are themselves groups with respect to the operation $\circ$. One very obvious one is $\{f_1, f_2, f_3, f_4\}$. Another is $\{f_1, g\}$.

Among the most elementary properties of groups are (1) that each group has an identity element--that is, an element i such that, for each member g of the group, $g \circ i = g$, and (2) that each member g of the group has an inverse--that is, an element g^{-1} such that $g \circ g^{-1} = i$. In fact, these two properties together with associativity constitute another characterization of the group concept. For the eight-membered group G, i is f_1, and the inverses of the members of G can be found from the table. Notice that, even though G is not commutative, for each $g \in G$, $g^{-1} \circ g = g \circ g^{-1} = i$. Notice also that, for elements g and h of G, $[g \circ h]^{-1} = h^{-1} \circ g^{-1}$. Both of these properties hold of all groups. For the examples given above in which the group operation is addition of real numbers, i is 0 and the inverse of each element is its opposite. For those in which the group operation is multiplication of real numbers, i is 1 and the inverse of each element is its reciprocal.

The group concept is one of the major unifying concepts in mathematics, and there is an extensive literature on the subject. You may find interesting the article "To Teach Modern Algebra" by Carl H. Denbow in the March, 1959 issue of The Mathematics Teacher, and the book The New Mathematics, by Irving Adler, published by John Day.

Q. 1. (a) Yes; $\{(x, y) : y = x - 5\}$ (b) Yes; $\{(x, y) : y = -x\}$

(c) Yes; $\{(x, y) : y = -x\}$ [or: $\{(x, y); x \leq 2 : y = -x\}$]

(d) No [$\mathscr{D}_h \not\subseteq \mathscr{D}_g$]

(e) Yes; $\{(x, y) : y = 3x - 8\}$ [or: $\{(x, y), x \geq \frac{3}{4} : y = 3x - 8\}$]

2. (a) Yes

(b) No [There are two arguments x_1 and x_2 of both g and h such that $g(x_1) = g(x_2)$ but $h(x_1) \neq h(x_2)$.]

(c) No [$\mathscr{D}_h \not\subseteq \mathscr{D}_g$] (d) Yes (e) Yes (f) Yes

R. 1. (a) (Don, 11), (Mike, 20), (Sally, 11)

(b) (Don, 30), (Mike, 99), (Sally, 28)

(c) (Don, 17), (Mike, 29), (Sally, 15)

(d) (Don, 16), (Mike, 31), (Sally, 18)

(e) (Don, 1), (Mike, −2), (Sally, −3)

(f) (Don, 11), (Mike, −40), (Sally, −33)

(g) (Don, 11), (Mike, −40), (Sally, −33)

2. (a) 28 (b) −32 (c) −28 (d) −2

(e) 88 (f) 217 (g) 52 (h) 217

(i) $3(7x + 3) - 5$ [or: $21x + 4$] (j) $(3x - 5)(7x + 3)$

(k) $(3x - 5)^2$ (ℓ) $(7x + 3)^2$

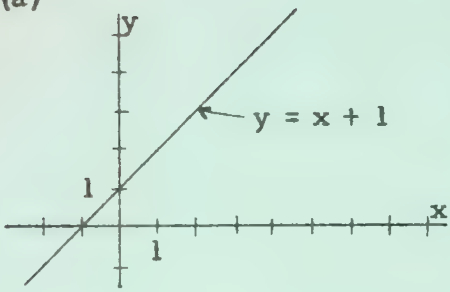
(m) $58x^2 + 12x + 34$ (n) $-40x^2 - 72x + 16$

3. (a) Yes (b) Yes (c) Yes

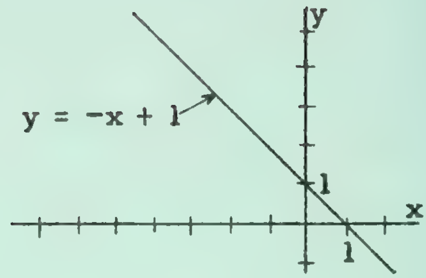
(d) No [$B(1) \neq f(A(1))$] (e) Yes

(f) No [A(0) is not in the domain of m, so 0 is not in the domain of the function $m \circ A$.]

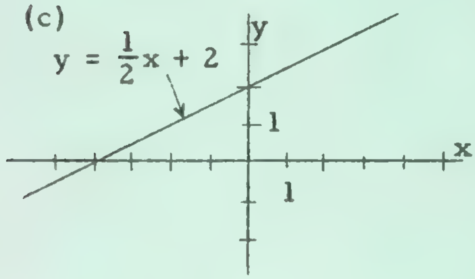
S. 1. (a)



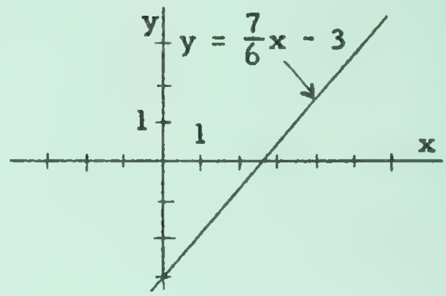
(b)



(c)



(d)



2. (a), (i), (ℓ), (t), (b), (g), (j), (p), (q),
(c), (f), (k), (o), (s), (d), (e), (h), (m), (n), (r)

3. (a) $\{(x, y): y = 7x - 26\}$ (b) $\{(x, y): 5x + 8y = 0\}$
(c) $\{(x, y): x + y = 5\}$ (d) $\{(x, y): 5y = 13x - 1\}$

4. (a) (1, 8) (b) (2, 10) (c) (5, 19) (d) (7, 1)
(e) (2, 6) (f) (3, 11) (g) (5, 20) (h) (4, 3)
(i) (6, 2) (j) $(\frac{3}{23}, \frac{67}{23})$

[In discussing the parts of Exercise 4 bring out that (a) can be most easily solved by first solving '$3x + 5 = 5x + 3$', and that (b), (c), and (d) can be treated similarly. Also, part (e) is easily solved by "adding and subtracting": $2y = 12$, $2x = 4$. (f) and (h) can be treated similarly.]

5. (a) $\{(x, y): y = 4x - 7\}$ (b) $\{(x, y): y = 5x + 4\}$

(c) $\{(x, y): y = 3x + 9\}$ (d) $\{(x, y): y = 2(7x - 1)\}$

(e) $\{(x, y): y = \frac{x}{2} + \frac{11}{2}\}$ (f) $\{(x, y): y = \frac{x}{4} - 2\}$

(g) $\{(x, y): y = x\}$ (h) $\{(x, y): y = -\frac{1}{2}x\}$

[Part (a) of Exercise 5 can be solved like the Sample. Here is another method: For any nonzero value of 'm' the equation '$y - 1 = m(x - 2)$' defines a linear function which contains (2, 1). The value of 'm' is the slope of the linear function. So, an answer for (a) is: $\{(x, y): y - 1 = 4(x - 2)\}$. The same method [or that of the Sample] can be used for part (b), once one sees that he is concerned with a linear function having slope 5. An alternative is to note that '$y - 5x = 14 - 5 \cdot 2$' defines a linear function of slope 5 which contains (2, 14). For part (c), one may find the slope [3] and arrive at once at '$y = 3x + 9$'. For part (h), students may graph '$y = 2x$' on squared paper, draw the line perpendicular [using a protractor, perhaps] to this through the graph of the origin, and notice that this line goes through the graph of, for example, (−2, 1).]

6. (a) $a(cx + d) + b$ (b) $c(ax + b) + d$

(c) $(ax + b)(cx + d)$ (d) $(cx + d)(ax + b)$

7. (a)

3	8
4	11
5	14
6	17
10	29
15	44

(b)

5	7
7	15
7.5	17
8	19
8.3	20.2
10	27

(c)

2	−3
8	−1
9	$-\frac{2}{3}$
13	$\frac{2}{3}$
−5	$-\frac{16}{3}$
−7	−6

> Correction. On page 5-271, in line 13, there should be a '?' after 'second'.

T. 1. 125 2. $\frac{5}{3}$ 3. $\frac{25}{3}$ 4. \$85 5. (c)

6. (a) $\frac{3}{4}$ (b) 20 (c) $\frac{7}{2}$ (d) 144

(e) 35 (f) 72 (g) 9 (h) 15

(i) 4 (j) 20 (k) 3 (ℓ) 7

7. 68¢ 8. $2\frac{7}{9}$ 9. $12\frac{1}{2}$ 10. 24.5 feet 11. (b)*

12. 8 13. 4 14. $\frac{10}{3}$ 15. 25

16. (a) $C = kr$ (b) $s = kP$ (c) $R = k\ell$ (d) $V = kT$

(e) $RA = k$ (f) $nc = k$ (g) $Id^2 = k$ (h) $f = \frac{k\sqrt{F}}{\ell}$

17. r is multiplied by 5 18. 4×402.5 [or: 1610]; 338.1

19. 25.2 20. $\frac{9}{4}$ feet 21. $\frac{pbc}{aq}$ ☆ 22. 14.5

☆ 23. $9\sqrt{2}$ centimeters

---

* If $M/N = 4$ and if, for each argument e of N such that $N(e) = 0$, $M(e) = 0$, then M varies directly as N. Compare with Exercises 5 and 6 of Part C on page 5-155.

$\underline{U}$. 1. (a) $(x + \frac{7}{2})^2 - \frac{69}{4}$ (b) $(x + 4)^2 - 21$ (c) $(x + \frac{9}{2})^2 - \frac{89}{4}$

(d) $(x - 5)^2 - 24$ (e) $(x - \frac{3}{2})^2 + \frac{11}{4}$ (f) $(x + 1)^2 + 16$

(g) $(x + 2)^2 - 11$ (h) $(x - \frac{5}{2})^2 - \frac{17}{4}$ (i) $3(x - 1)^2 + 6$

(j) $12(x - \frac{1}{12})^2 + \frac{59}{12}$

2. (a) $x = -\frac{7}{2}$ (b) $x = -4$ (c) $x = 1$

(d) $x = \frac{5}{2}$ (e) $x = -\frac{1}{4}$ (f) $x = \frac{7}{10}$

3. (a) $(0, 0)$; 6 (b) $(0, 0)$; $\sqrt{2}$ (c) $(2, 0)$; 5

(d) $(6, 0)$; 5 (e) $(-5, 0)$; 5 (f) $(0, 1)$; 5

(g) $(0, -7)$; 5 (h) $(2, 3)$; 5 (i) $(-4, 9)$; 5

(j) $(-4, 9)$; 5 (k) $(3, -4)$; 5 (ℓ) $(3, -4)$; 5

(m) $(6, 2)$; 5 (n) $(7, 1)$; 6 (o) $(-5, -2)$; 7

Correction. Delete Exercise 7(g) on page 5-274. It repeats Exercise 5 on page 5-184.

V. 1. (a) $4, 1$ (b) $2, 5$ (c) $-4, -2$ (d) $-5, -3$

(e) $-2, 3$ (f) $-3, 4$ (g) $-7, 0$ (h) $2, \frac{5}{2}$

(i) $-\frac{2}{3}, 3$ (j) $\frac{1}{2}, 1$ (k) $\frac{1}{2}, 2$

2. (a) $-1 - \sqrt{5}, -1 + \sqrt{5}$ (b) $-7, 3$ (c) $-2, 8$

(d) $2, 6$ (e) $-2, -1$ (f) $-6, 1$

(g) $1 - \sqrt{6}, 1 + \sqrt{6}$ (h) $-3 - \sqrt{13}, -3 + \sqrt{13}$

3. (a) $-8, 5$ (b) $-1, -\frac{2}{3}$ (c) $5 - \sqrt{10}, 5 + \sqrt{10}$

(d) $-\frac{3}{2}, 2$ (e) $\frac{2 - \sqrt{14}}{5}, \frac{2 + \sqrt{14}}{5}$

(f) $\frac{5 - \sqrt{57}}{4}, \frac{5 + \sqrt{57}}{4}$ (g) $\frac{7 - \sqrt{17}}{4}, \frac{7 + \sqrt{17}}{4}$

(h) $\frac{4 - \sqrt{6}}{2}, \frac{4 + \sqrt{6}}{2}$

4. (a) $-4.4, -0.4$ (b) $-0.6, 2.3$

(c) $-0.8, 3.8$ (d) $-2.6, 0.6$

5. (a) $(0, -4)$ (b) $(\frac{-5 - \sqrt{41}}{2}, 0), (\frac{-5 + \sqrt{41}}{2}, 0)$

(c) $(\frac{-5 - 3\sqrt{5}}{2}, 1), (\frac{-5 + 3\sqrt{5}}{2}, 1)$

(d) $(-6, 2), (1, 2)$ (e) $(-7, 10), (2, 10)$

6. (a) 3 (b) 10 (c) 3 (d) 18 (e) $x^2 + 3x - 10 = 0$

7. (a) 11 and 16 (b) $2\sqrt{269}$ (c) 14" by 14"

(d) 10 and 7, or -7 and -10 (e) 8 and 13, or -13 and -8

(f) 6.25 miles

W. 1. (a) $(8, 4)$ (b) $(9, 4)$ (c) $(-8, 2)$

(d) $(5, 1)$ (e) $(-3, 3)$ (f) $(5, 3)$

(g) $(1, -1)$ (h) $(-1, 1)$ (i) $(4, 5)$

(j) $(-1, -3)$ (k) $(-1, 3)$ (ℓ) $(-1, -4)$

(m) $(2, -1)$ (n) $(1, 3)$ (o) $(7, -2)$

(p) $(4, -1)$ (q) $(2, -\frac{1}{2})$ (r) $(3, 5)$

(s) $(-3, 6)$ (t) $(-2, 13)$

2. (a) $(2, -7, -\frac{3}{4})$ (b) $(\frac{1}{2}, -1, 3)$ (c) $(2, -\frac{1}{2}, \frac{1}{3})$

(d) $(2, \frac{1}{2}, -1)$ (e) $(\frac{2}{3}, -3, 3)$ (f) $(\frac{1}{2}, -\frac{1}{3}, \frac{1}{4})$

3. (a) $\{(9, -5), (-3, 1)\}$ (b) $\{(-4, -11), (1, -1)\}$

(c) $\{(0, -1), (2, 3)\}$ (d) $\{(-9, -6), (6, 9)\}$

(e) $\{(2, -$[illegible]$), (-2, 1)\}$ (f) $\{(\frac{3}{2}, \frac{5}{2}), (-1, -5)\}$

(g) $\{(8, -11), (2, 1)\}$ (h) $\{(7, 5), (0, -2)\}$

4. (a) \$1.40 [$3c + 2p = 69$, $2c + 3q = 71$]

(b) 13¢ [$d + 25r = 63$, $d + 50r = 113$]

(c) 5900 lbs. [$t + c = 8000$, $t + \frac{1}{3}c = 6600$]

(d) 5:00 p.m. [$f = s + 2$, $30f = 40s$]

(e) The data are inconsistent. [t...dollar-cost per square foot for the table top, ℓ...dollar-cost of the table legs; $\ell + 16t = 42$, $\ell + 25t = 72$. The solution of this system is $(-34/3, 10/3)$. But, we are seeking numbers of arithmetic; and, since there is no pair of positive numbers which is a solution of the system, there is no pair of numbers of arithmetic which satisfies the conditions of the problem.]

(f) The data are inconsistent. [To reduce the percentage of salt in the mixture, one should add water, not salt.]

(g) 1 inch [x...inch-height of 1 copy of Unit 1, y...inch-height of 1 copy of Unit 2; $8x + 3y = 12.5$, $4x + 5y = 11.5$]

(h) \$3, \$5 [$35r + 10v = 155$, $30r + 15v = 165$]

(i) 20 children, 10 adults [$25c + 60a = 1100$, $30c + 75a = 1350$]

(j) 17, 22 [$a = b + 5$, $a + b = 39$]

(k) 4 m.p.h., 36 m.p.h. [$3w + 2r = 84$, $2w + 3r = 116$]

(ℓ) 800, older; 900, newer [$3e + 4n = 6000$, $4e + 2n = 5000$]

(m) \$9000 [x...dollars invested at 3%, y...dollars invested at 4%; $.03x + .05y = 370$, $.04x + .04y = 360$. The second equation tells us that $x + y = 9000$. No need to solve the system.]